SOME NEW RESULTS CONCERNING THE ASYMPTOTIC STABILITY AND THE RESPONSE OF NONLINEAR SYSTEMS

A FINAL REPORT ON

"SUPPORTING RESEARCH STUDIES TO BOOSTER FLIGHT CONTROL PROBLEMS"

by

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I. SUMMARY

A sufficient condition given for the asymptotic stability of a continuous single monotone nonlinearity system with slope confined to $(0, k_2)$ having a transfer function $G(j\omega)$ is

$$\mathrm{Re}\ (1 + X(j\omega) + Y(j\omega) + \alpha j\omega)(G(j\omega) + 1/k_2) \geq 0$$

where α is a positive number, $x(t) \leq 0$ for $t \leq 0$ and zero for $t > 0$, $y(t) \leq 0$ for $t > 0$ and zero for $t < 0$ and $\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)dt < 1$. As is shown by examples the new criterion gives better results than existing criteria. Also developed are improved stability criteria for the case of the nonlinearity being an odd monotone function and for a nonlinearity with a monotone bound having a certain degree of symmetry.

A number of theorems giving bounds on the response of the single monotone nonlinearity system with initial condition and external excitation are presented. Under certain circumstances these bounds, which are useful in design, can also be employed to show Liapunov stability.

Improved time-frequency domain stability criteria are also developed for systems with a single time varying nonlinearity, for sample data systems with a single time invariant nonlinearity, and for continuous nonlinear systems having a number of nonlinearities.

II. INTRODUCTION

This work is based upon the observation that if two functions $\sigma(t)$ and $\phi(t)$ satisfy $\sigma\phi(\sigma) > 0$ for $\sigma \neq 0$, and $\phi(\sigma)$ is a monotone increasing function of σ, a bound can be placed upon $\int_0^T [\sigma(t) * A(t)]\phi(t)dt$ in terms of $\int_0^T \sigma(t)\ \phi(t)dt$ provided that A(t) satisfies certain conditions. This relationship is used in Chapter I to give improved conditions for the asymptotic stability of a continuous time invariant system with a single monotone nonlinearity. A modification of this proof results in two other theorems, one for the asymptotic stability of a system with a single odd monotone nonlinearity and the other for a system with a nonlinearity having a monotone bound. In Chapter II bounds are obtained on the response of the continuous system whose stability was discussed in Chapter I. In addition to giving bounds on the response with an initial condition excitation, bounds are also developed on the response for an external input that is Fourier transformable in a finite time interval. If the input is itself bounded, these theorems permit the showing of Liapunov stability. Chapter III extends the results of Chapter I to systems having a single time varying nonlinearity, sampled data systems with a single nonlinearity, and continuous systems with a number of time invariant nonlinearities.

III CHAPTER I. THE STABILITY OF SINGLE NONLINEARITY CONTINUOUS SYSTEMS

A. Introduction

This chapter deals mainly with sufficient conditions for the asymptotic stability in the large of the system shown in Figure 1 with $\phi(\sigma)$ a continuous monotone increasing nonlinearity. Several recent works have considered this problem [1-4]. Reference [4] by one of the authors concerns a part of the research presented in this report, namely corollary 3 of theorem 1.1 and a simplified version of theorem 1.2. Brockett and Willems [3] presented a sufficient condition for the asymptotic stability of this system with the nonlinearity being a continuous monotone function. With $0 \leq d\phi/d\sigma \leq k_2$, it was shown that

$$\mathrm{Re}[Z(j\omega) \quad (G(j\omega) + 1/k_2)] \geq 0$$

is a sufficient condition for asymptotic stability where Z is either a physically realizable RL driving point impedance function or its reciprocal. Z allows the angle of $G(j\omega) + 1/k_2$ to lie outside the $\pm 90^{\circ}$ band in only one direction. In other words, the polar plot of $G + 1/k_2$ is restricted to lie in three quadrants. The present work presents a theorem for the monotone nonlinearity which permits a larger class of Z multipliers to be used, thereby allowing $G + 1/k_2$ to lie in four quadrants. The same approach is applied to give improved conditions for the asymptotic stability of a system with a single odd monotone nonlinearity and for a system with a nonlinearity having a monotone bound.

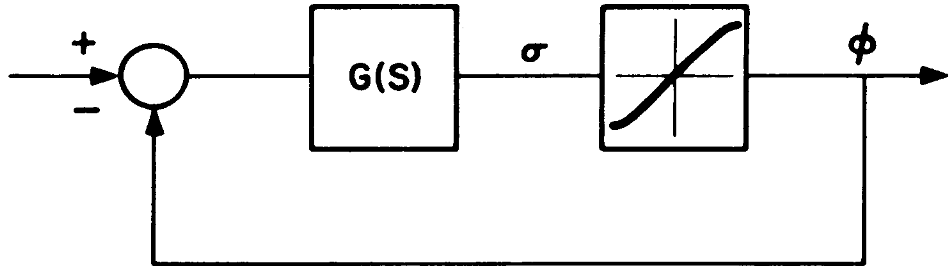

Figure 1. The continuous system under consideration.

In using the following theorems, if the nonlinear characteristic satisfies $k_1|\sigma| < |\phi(\sigma)|$, the linear transformation $\phi_1(\sigma) = \phi(\sigma) - k_1\sigma$ giving $G_1(s) = G(s)/(1 + k_1G(s))$ should first be carried out, provided that in the case of theorem 1.1 $\phi_1(\sigma)$ is a monotone increasing function. The theorems are then applied to the transformed system with nonlinear characteristic $\phi_1(\sigma)$ and transfer function $G_1(s)$.

In the following work the notation $\phi(\sigma)$ is used when the properties of the nonlinearity are under consideration and $\phi(t)$ is used when the time varying variable $\phi(\sigma(t))$ is being discussed.

B. A Theorem For Monotone Nonlinearities

Theorem 1.1

For the system shown in figure 1 let the following hold:

a. $0 \leq d\phi(\sigma)/d\sigma \leq k_2$ where k_2 is a positive number, both $\phi(\sigma)$ and $\sigma - \phi(\sigma)/k_2 = 0$ only for $\sigma = \phi(\sigma) = 0$, and $d\phi(\sigma)/d\sigma$ be a continuous function of σ.

b. $G(s) = N(s)/D(s)$ with the degree of $N(s)$ at least one less than the degree of $D(s)$ and with the zeros of $D(s)$ in the left half s plane. $N(s)$ and $D(s)$ are assumed to have no common factors in the right half s plane or on the $j\omega$ axis.

c. $\lim_{|\sigma| \to \infty} \int_0^{\sigma} \phi(\sigma)d\sigma/|\phi(\sigma)|^2 = \infty$ or

$\lim_{|\sigma| \to \infty} |\phi(\sigma)| = h|\sigma|$ where $h > 0$.

Then a sufficient condition for asymptotic stability in the large is that

$$\mathrm{Re}[Z(s)(G(s) + 1/k_2)] \geq 0 \tag{1.1}$$

for $s = j\omega$ for all real ω where

$$Z(s) = 1 + \alpha s + X(s) + Y(s). \tag{1.2}$$

The time function $x(t) = 0$ for $t > 0$ and $y(t) = 0$ for $t < 0$. Both of these functions are assumed to be the sum of a piecewise continuous function which is Fourier transformable and shifted impulse functions that satisfy

$$\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)\, dt < 1 \tag{1.3}$$

with both $x(t)$ and $y(t) \leq 0$. The magnitude of the piecewise continuous component of $x(t)$ is assumed to be less than $\ell \exp(ft)$ where ℓ and f are positive numbers. The contribution of the impulses to the integral is to be taken as the strengths of the impulses. α is a positive number.

Corollary 1. In addition to the conditions of theorem 1.1, if $\phi(\sigma)$ is an odd monotone nonlinearity that is, if $\phi(\sigma) = -\phi(-\sigma)$, the assertion of the theorem holds with (1.3) becoming

$$\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)\, dt < 1 \text{ where } x(t) \text{ and } y(t) \text{ are}$$

permitted to take on positive as well as negative values.

Corollary 2. If $G(s)$ has poles on the $j\omega$ axis, $G(s)$ is required to be stable in the limit; that is, for an arbitrarily small positive number

ε, the zeros of $1 + \varepsilon G(s)$ must all be in the left half s plane. Also, the slope condition becomes $\geq \delta > 0$ and (1.1) becomes $\geq \delta_2 > 0$ where δ and δ_2 are small positive numbers. The other conditions are unchanged.

Corollary 3. If c is not satisfied, the assertion of the theorem holds with x(t) required to be identically zero.

Since the statement of the theorem is somewhat involved, a discussion of its various conditions is in order. The slope bound condition a includes a requirement that $d\phi(\sigma)/d\sigma$ be a continuous function of σ whose purpose is to insure the Fourier transformability and piecewise continuity of $\sigma(t)$, $\dot{\sigma}(t)$, and $\phi(t)$; any other property insuring this result would suffice. Condition b is used to guarantee that if $\sigma(t)$ and $\phi(t)$ are bounded for all t and approach zero as $t \to \infty$, the other state variables of the system have this same type of behavior. In addition, having the degree condition holding allows the αs term to be used in the frequency domain criterion since it insures the Fourier transformability of that component of $d\sigma(t)/dt$ due to $-\phi(t)$. The first part of condition c permits the nonlinear characteristic to have a behavior which ranges from that of a saturation function to a linear characteristic for large values of σ, with the first mentioned function being allowed but not the second. The second part of this condition permits a linear characteristic.

C. Application of the Theorem

In applying the theorem it is convenient to first draw the log magnitude and phase plots of $G(j\omega) + 1/k_2$. Since $|G(j\omega)|$ approaches zero for ω sufficiently large, above a certain frequency, ω_c, $|G(j\omega)| < 1/k_2$, and hence the phase angle of $G(j\omega) + 1/k_2$ will be less than 90^o. The real part condition will be satisfied with $Z(s) = 1$ for $\omega \geq \omega_c$. If it is also satisfied for $\omega < \omega_c$, asymptotic stability will be guaranteed.

If the real part condition is not satisfied for $\omega < \omega_c$, a frequency varying $Z(j\omega)$ must be chosen in an attempt to show stability. Since the real part condition is already satisfied for $\omega > \omega_c$, $Z(j\omega)$ should not disturb this property. The general philosophy to be followed in searching for a suitable $Z(j\omega)$ function is to observe the frequency bands in which the angle of $G(j\omega) + 1/k_2$ lies outside the $\pm 90^o$ band and then to try to choose a $Z(j\omega)$ function such that its phase angle when added to that of $G(j\omega) + 1/k_2$ gives a resultant phase angle which lies within the $\pm 90^o$ band.

D. Two Z(s) Multipliers

1. $\prod_{i=1}^{n} \left(\frac{s + a_i}{s + b_i}\right) \prod_{j=1}^{m} \left(\frac{s - c_j}{s - d_j}\right) + \alpha s$

$$0 < a_1 < b_1 < a_2 < b_2 < \ldots a_n < b_n$$

$$0 < c_1 < d_1 < c_2 < d_2 < \ldots c_m < d_m$$

The first product is an RL impedance function and the second is transformed into an RL impedance function by replacing s with -s. Therefore, the poles and zeros of the first product alternate along the negative real axis while the critical points of the second product are along the positive real axis of the s plane. Expressing this function in a partial fraction expansion gives

$$1 - \sum_{i=1}^{n} \frac{f_i}{s + b_i} - \sum_{j=1}^{m} \frac{l_j}{-s + d_j} + \alpha s$$

where f_i and l_j are positive numbers. Since the partial fraction expansion coefficients are negative for both the left half and right half plane poles, the time function corresponding to these poles is non-positive. Using $F(0) = \int_{-\infty}^{+\infty} f(t)dt$, where $F(j\omega)$ is the Fourier transform of f(t), in conjunction with

$$\prod_{i=1}^{n} \left(\frac{s + a_i}{s + b_i}\right) \prod_{j=1}^{m} \left(\frac{s - c_j}{s - d_j}\right) - 1$$

gives

$$\prod_{i=1}^{n} \left(\frac{a_i}{b_i}\right) \prod_{j=1}^{m} \left(\frac{c_j}{d_j}\right) - 1 \text{ as the area associated with } x(t) + y(t)$$

for this Z(s). Since these time functions are non-positive and the magnitude of this area is less than 1, the given function is an allowed one for general monotone nonlinearities.

The phase characteristic of this function is more flexible than the Z(s) multipliers considered in [4] because it is possible to switch back and forth from a leading to a lagging function or vice versa if desired. A typical phase angle plot is shown in Figure 2 for the particular case $n = 2$, $m = 2$. It is to be noted that the magnitude of the angle can approach 90° as closely as desired.

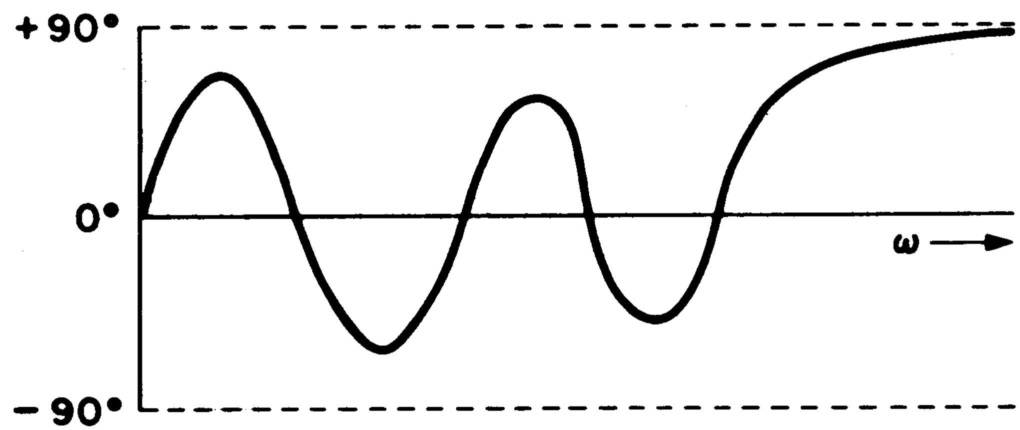

Figure 2. Angle plot for a particular type 1 function.

Example 1. Brockett and Willems [3] indicated that

$$G(s) = \frac{s^2}{s^4 + a\,s^3 + b\,s^2 + c\,s + d}$$

with a, b, c, and d chosen such that the poles of G(s) lie in the left half s plane was a worthwhile function for future study since their criterion did not apply to it. This G(s) is to be considered assuming that k_2 is large but finite with the nonlinearity required to be monotone. An angle plot of $G(j\omega) + 1/k_2$ is given in Figure 3. Let $Z(s) = (-s + p)(s + r)/(-s + q)$ with $p < q$. Division of the numerator by the denominator shows that this Z(s) belongs to the function 1 class with $n = 0$, $m = 1$. The reason for this choice of $Z(j\omega)$ is that its angle lags at low frequency and leads at high frequency, which is the required behavior if the angle of the product function is to lie within the $\pm 90^o$ band. The variation in angle for $G(j\omega) + 1/k_2$ at low frequency can be handled by choosing p sufficiently small. However, a problem is encountered in following the variation from near + 180 to $- 180^o$. First, let G(s) have four real poles located at $- a_1$, $- a_2$, $- a_3$, and $- a_4$. Then $Z(s)(G(s)+1/k_2)$ is with $s = j\omega$

$$\frac{(-s + p)}{(-s + q)}\,(s + r)\;\left[\frac{s^2}{(s + a_1)(s + a_2)(s + a_3)(s + a_4)}\right] + \frac{R(s)}{k_2}$$

where R(s) is the even part of Z(s). The angle of the first term above with $q = a_1$ and $r = a_2$ is

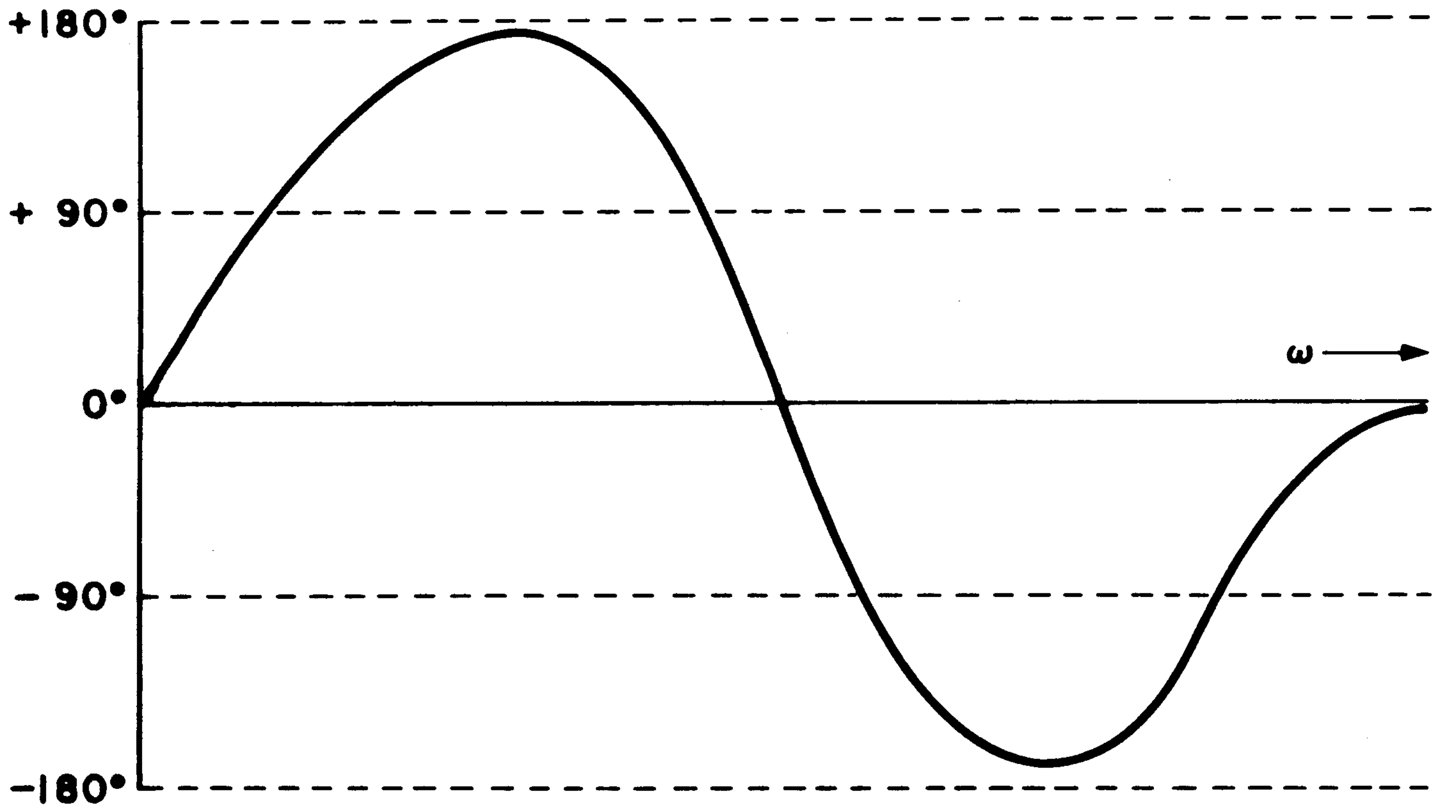

Figure 3. Angle plot of $G(j\omega) + 1/k_2$ for example 1.

$$180^\circ - \tan^{-1}\frac{\omega}{p} - \tan^{-1}\frac{\omega}{a_3} - \tan^{-1}\frac{\omega}{a_4}.$$

The value of p can be chosen small enough such that at low frequency when the magnitude of the first term is equal to $R(j\omega)/k_2$ which itself is positive, the above angle is less than 90°. Since (1.1) is satisfied, asymptotic stability in the large is guaranteed.

Next, consider the case of $G(s) = s^2/((s^2 + 2\zeta\omega_m s + \omega_m^2) \times (s + a_1)(s + a_2))$ where $\zeta < 1$ and $\omega_m > 0$. The angle of $Z(j\omega)\ G(j\omega)$ in this case with q and r chosen equal to a_1 and a_2 respectively is

$$180^\circ - \tan^{-1}\frac{\omega}{p} - \tan^{-1}\frac{2\zeta\omega_m}{(\omega_m^2 - \omega^2)}.$$

As before, a suitable choice of p will make the angle of $Z(j\omega)(G(j\omega) + 1/k_2)$ lie in the $\pm 90^\circ$ band for all ω and asymptotic stability in the large has been shown.

Finally, let $G(s) = s^2/(s^2 + 2\zeta\omega_m s + \omega_m)^2$. The angle of $Z(j\omega)\ G(j\omega)$ is, with $r = q = \omega_m$,

$$180^\circ - \tan^{-1}\frac{\omega}{p} + 2\tan^{-1}\frac{\frac{\omega}{\omega_m}\left[1 - 2\zeta - \left(\frac{\omega}{\omega_m}\right)^2\right].}{\left[1 - (1 - 2\zeta)\left(\frac{\omega}{\omega_m}\right)^2\right]}.$$

If $\zeta > .5$, and p suitably chosen, (1.1) is satisfied and asymptotic stability in the large is demonstrated. This Z(s) will not satisfy (1.1) for $\zeta < .5$ and hence no information is available on the stability of the system. Fitts [5] has shown that periodic solutions exist with $\phi(\sigma) = \sigma^3$ and $\zeta = .01$ for the 2 pair complex conjugate pole case. The author has obtained steady state oscillations with $\phi(\sigma)$ an odd saturation function for $\zeta = .045$ and with $\phi(\sigma)$ an unsymmetrical saturation nonlinearity for $\zeta = .075$.

In summary, with a monotone nonlinearity asymptotic stability in the large can be guaranteed for the given G(s) if the poles are all real, if two are real and the other two complex, or if all four are complex provided that $\zeta > .5$.

2. $1 + \sum_{i=1}^{n} a_i \exp(b_i s) + \alpha s$ with the b_i's being real numbers,

$\alpha > 0$, and $\sum_{i=1}^{n} |a_i| < 1$. If all the a_i's are negative, this multiplier can be used for a general monotone nonlinearity but if some are positive, the nonlinearity must be an odd function. The angle of this Z(s) is $\tan^{-1}\left(\left(\sum_{i=1}^{n} a_i \sin b_i\omega + \alpha\omega\right)/\left(1 + \sum_{i=1}^{n} a_i \cos b_i\omega\right)\right)$. This multiplier is capable of providing a rapid change in phase shift from near -90° to $+90^{\circ}$, but the periodic nature of the exponential part of this function can make it a difficult one to work with. A useful special case results when $\sum_{i=1}^{n} a_i \exp(b_i j\omega) =$ $j \sum_{i=1}^{n/2} 2a_i \sin b_i\, \omega$ with $\sum_{i=1}^{n/2} 2|a_i| < 1$. The angle of $Z(j\omega)$ for this case is $\tan^{-1}\left(\sum_{i=1}^{n/2} 2a_i \sin b_i\, \omega + \alpha\, \omega\right)$ which is simpler than the general result. On the other hand, the angle variations in the function are constrained; if α is a very small number, the angle lies in a $\pm$ 45° band at low frequencies. The use of this class of multiplier is illustrated by the following example.

Example 2. Dewey and Jury [2] considered the case of $G(s)=40/s(s+1)(s^2+.8s+16)$ using their criterion for monotone nonlinearities and showed stability for nonlinearities having a slope restricted to $(\varepsilon, 1.43)$. The system is stable for linear gains in the sector $(\varepsilon, 1.76)$. Because G(s) has a pole on the $j\omega$ axis, corollary 2 must be applied rather than the theorem. From the root locus plot for $1 + \varepsilon G(s)$, where ε is a small positive number, it is seen that G(s) is stable in the limit. From the Figure 4 plot of the angle of $G(j\omega) + 1/1.76$, the angle lies outside the $\pm$ 90° band in the frequency ranges 0-2.75 and 2.97-3.75, lagging in the former case and leading in the latter. Although the peak deviation outside the $\pm$ 90° band is only 36° in the lagging direction and 16° in the leading direction, the peak slope of the angle in making the transition from outside the $\pm$ 90° band to the inside is about 40°/radian, making it impossible to use a Z(s) of the function 1 class. The magnitude of the slope of a Z(s) function belonging to the type 1 class is less than or equal to the slope of the angle of the double pole function $2 \tan^{-1} \omega/a$ which is $2a/(\omega^2 + a^2)$. For $\omega = a = 3$, approximately the values which would have to be chosen in attempting to use the function, the slope would be about 20°/radian, less than half the required value. Therefore, a function of the type 2 class is chosen in an effort to show asymptotic stability. Since the required angle for Z(s) is less

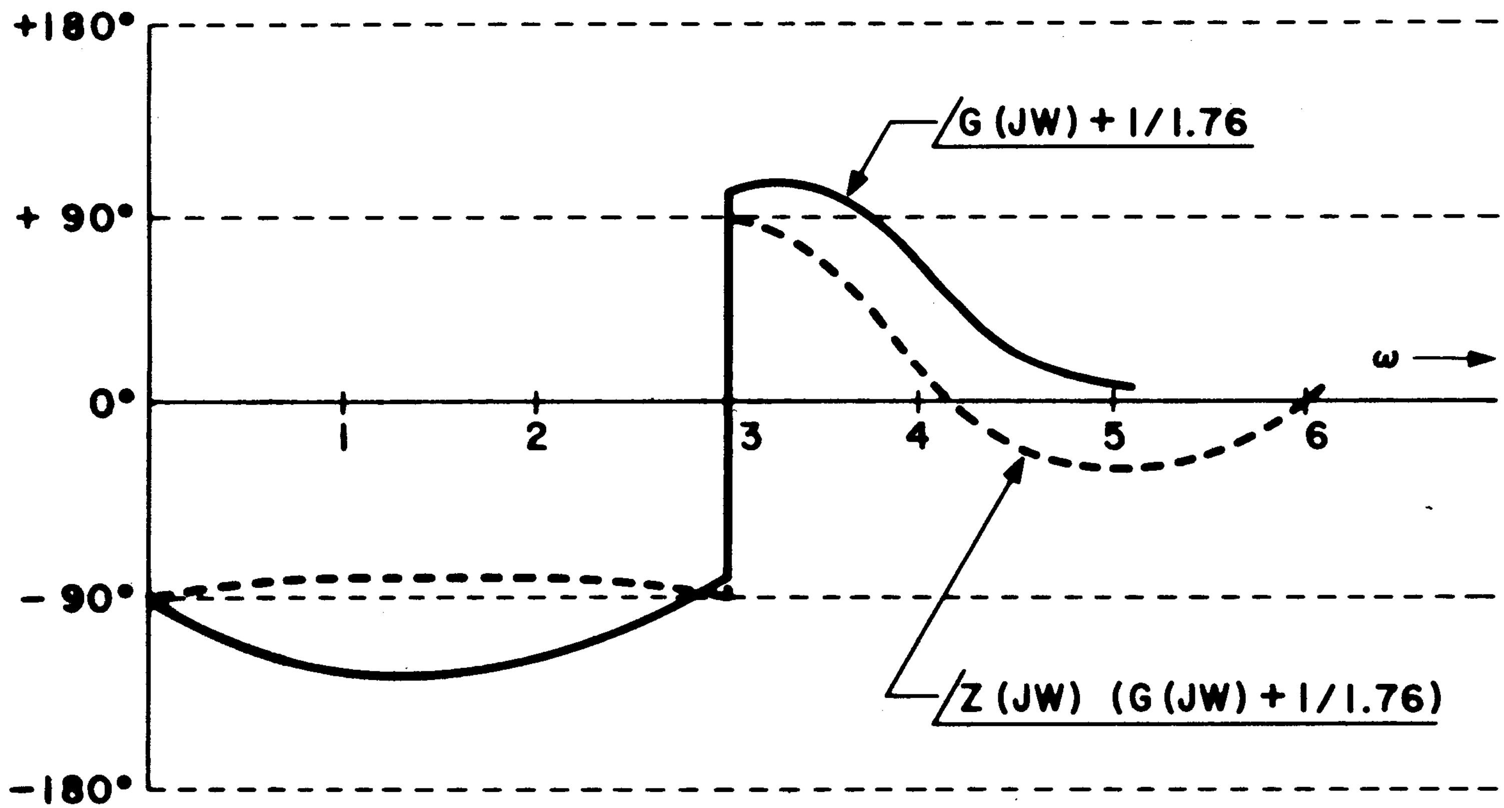

Figure 4. Angle plots pertinent to the first part of example 2.2.

than 45^{o}, and a leading angle followed by a lagging angle is required, $Z(j\omega)$ was chosen equal to $1 + j.999 \sin 1.118\omega + 10^{-10} j\omega$. Comparing this function with the time domain condition (1.3) shows that $\phi(\sigma)$ is required to be an odd monotone function. The 1.118 coefficient was picked to give an angle for $Z(j\omega)(G(j\omega) + 1/1.76)$ at $\omega = 2.98$, the frequency at which a zero occurs on the $j\omega$ axis for $G(j\omega) + 1/1.76$, of $\pm 90^{o}$. The amplitude of the sine term was chosen close to 1 to give a large change in angle while still satisfying the integral condition (1.3) and the 10^{-10} coefficient was chosen so that the $\alpha\, j\omega$ term does not come into play at low frequencies. The slope of the angle of this multiplier at $\omega = 3$ is about 60^{o}/radian. The plot of the angle of the product function also given in Figure 4 shows that the angle always remains within the $\pm 90^{o}$ except for $\omega = 0$, 2.98, and ∞ at which frequencies the angle magnitude is 90^{o}. Calculation of the real part of the product function at $\omega = 0$ gives .738. If $k_2 < 1.76$, the real part of the product is positive at $\omega = 2.98$. At ∞, this quantity is 1/1.76. Therefore, since (1.1) is satisfied with an inequality sign, all the conditions of corollaries 1 and 2 are satisfied and asymptotic stability in the large is guaranteed for slopes in the sector (ε, 1.76) for $\phi(\sigma)$ equal to an odd monotone nonlinearity.

In order to find an enlarged sector of assured asymptotic stability for the general monotone nonlinearity, $Z(j\omega) = 1 - .95 \exp(-1.045j\omega) + 10^{-10} j\omega$ was chosen for use with $G(j\omega) + 1/1.7$. The reasons for the choice of this function and the parameters for this case are identical with those of the previous case except that the coefficient of the exponential was chosen to give a zero phase shift for $Z(j\omega)$ in the middle of the transition region for the angle of $G(j\omega) + 1/1.7$. The slope of the angle of this $Z(j\omega)$ at $\omega = 3$ is about 30°/radian. Therefore, k_2 was reduced to 1.7 when it was found to be impossible to satisfy (1.1) with the given form of $Z(j\omega)$ and $k_2 = 1.76$. Figure 5 gives the pertinent plots for this example which show that the angle of $Z(j\omega)(G(j\omega) + 1/1.7)$ is in the $\pm 90^{\circ}$ band. At $\omega = 0$ the angle of the product is -90° but the real part is 2.38 while at $\omega = \infty$ the angle is 90° with the real part being $(1-.95\cos 1.045\omega)/1.7$. Therefore, the conditions of corollary 2 are satisfied and asymptotic stability in the large is guaranteed for the general monotone nonlinearity with slope in the sector $(\varepsilon, 1.7)$.

E. Proof of Theorem 1.1

Let the system be excited by initial conditions. The assumptions on $G(s)$ and on $\phi(\sigma)$ are sufficient to insure the continuity and Fourier transformability of $\sigma(t)$, $\dot{\sigma}(t)$, and $\phi(t)$ on any finite time interval [6] - [9]. Use will be made of these properties at several points in the proof. First, it will be shown that

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * (\sigma^n(t) - \phi^n(t)/k_2))\phi^n(t)\,dt =$$

$$c(T_n) \int_0^{T_n} (\sigma^n(t) - \phi^n(t)/k_2)\,\phi^n(t)\,dt \qquad (1.4)$$

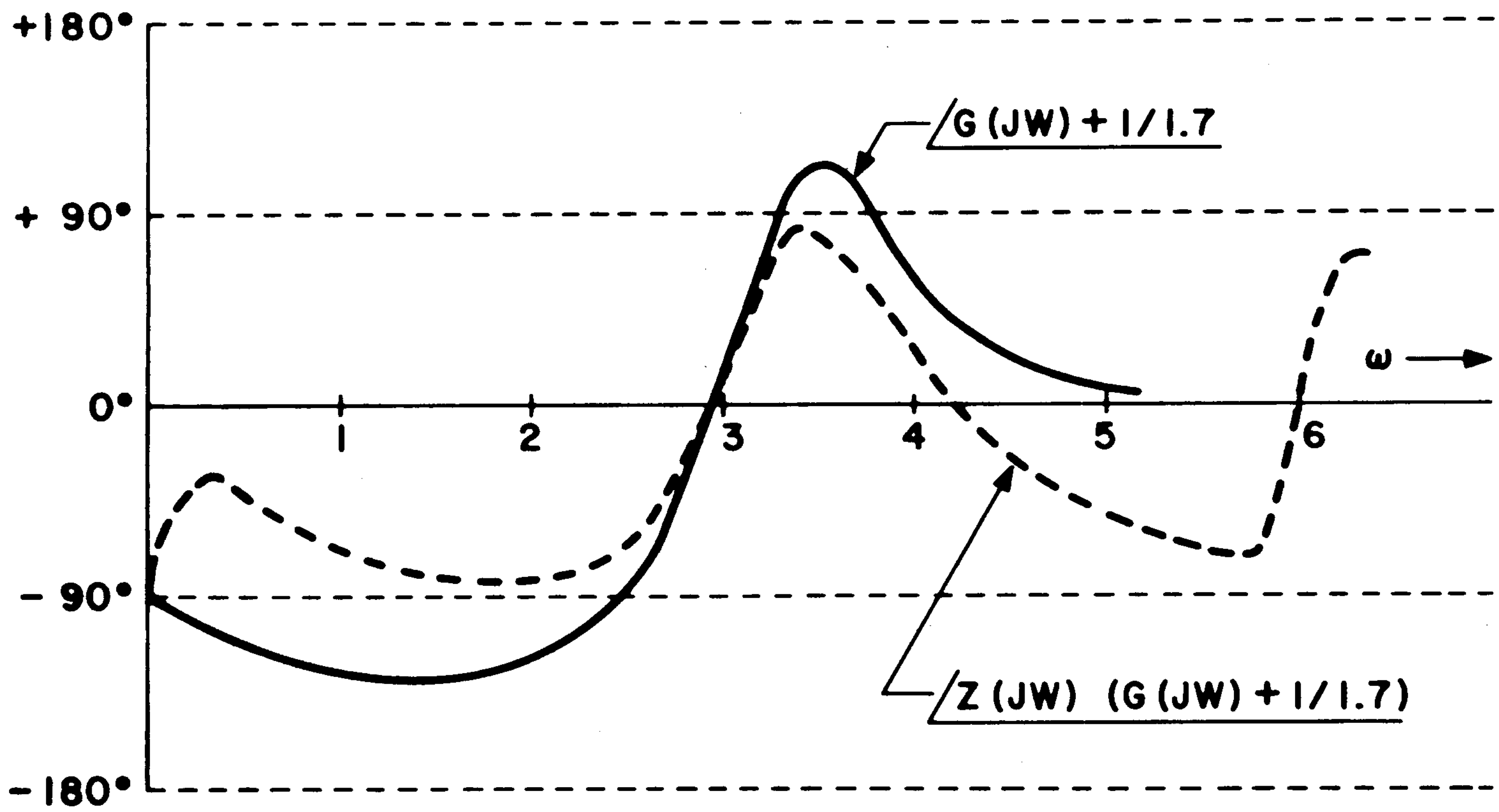

Figure 5. Angle plots for the second part of example 1.2.

where $c(T_n)$ is a positive number and * denotes convolution. The variables $\sigma^n(t)$ and $\phi^n(t)$ are equal to their non-superscripted counterparts in $(0, T_n)$ and zero outside this interval. Let $x'(t)$ and $y'(t)$ denote $x(t)$ and $y(t)$ respectively with the impulses removed. The integral involving these functions on the left hand side of (1.4) is given by

$$\int_0^{T_n} \int_{-\infty}^{0} x'(\lambda)\, (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\phi^n(\lambda)\, d\lambda\, dt +$$

$$\int_0^{T_n} \int_0^{\infty} y'(\lambda)\, (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\, \phi^n(t)\, d\lambda\, dt \qquad (1.5)$$

since $x'(\lambda) = 0$ for $\lambda > 0$ and $y'(\lambda) = 0$ for $\lambda < 0$. Because the primed functions, $\sigma^n(t)$ and $\phi^n(t)$ are continuous functions of t, and the integrand is non-zero over only a finite interval of time, the order of integration may be interchanged [10] to give

$$\int_{-\infty}^{0} x'(\lambda) \int_0^{T_n} (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\, \phi^n(t)\, dt\, d\lambda +$$

$$\int_0^{\infty} y'(\lambda) \int_0^{T_n} (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\, \phi^n(t)\, dt\, d\lambda. \qquad (1.6)$$

With the impulsive component of $x(t)$ given by $\sum_{i=1}^{u} a_i\, \delta(t + b_i)$ and that of $y(t)$ by $\sum_{j=1}^{v} c_i\, \delta(t - d_i)$, where a_i, b_i, c_i, and d_i are positive numbers, their contribution to the left hand side of (1.4) is

$$\sum_{i=1}^{u} a_i \int_0^{T_n} (\sigma^n(t+b_i) - \phi^n(t+b_i)/k_2)\, \phi^n(t)\, dt +$$

$$\sum_{j=1}^{v} c_i \int_0^{T_n} (\sigma^n(t-d_i) - \phi^n(t-d_i)/k_2)\, \phi^n(t)\, dt\, . \qquad (1.7)$$

Appearing in both (1.6) and (1.7) is an integral of the form

$$I(T) = \int_0^{T_n} (\sigma^n(t - T) - \phi^n(t - T)/k_2)\ \phi^n(t)\ dt \tag{1.8}$$

where T is a real number. At this point a positive bound will be developed on (1.8). Let (1.8) be rewritten as

$$\begin{aligned} I(T) = & \int_0^{T_n} (\sigma_+^n(t - T) - \frac{\phi_+^n(t - T)}{k_2})\ \phi_+^n(t)\ dt\ + \\ & \int_0^{T_n} (\sigma_-^n(t - T) - \frac{\phi_-^n(t - T)}{k_2})\ \phi_-^n(t)\ dt\ + \\ & \int_0^{T_n} (\sigma_+^n(t - T) - \frac{\phi_+^n(t - T)}{k_2})\ \phi_-^n(t)\ dt\ + \\ & \int_0^{T_n} (\sigma_-^n(t - T) - \frac{\phi_-^n(t - T)}{k_2})\ \phi_+^n(t)\ dt \end{aligned} \tag{1.9}$$

where the + and - subscripts refer to the positive and negative values of the associated functions, respectively: as an example $\phi_+^n(t)$ is equal to $\phi^n(t)$ when $\phi^n(t) > 0$ and zero otherwise. The lemma may be applied to the first two integrals since $\sigma^n(t)$ and $\phi^n(t)$ are continuous functions of time that are zero outside $(0, T_n)$ the two functions forming the integrand of both integrals are non-negative and non-positive respectively, and

$$d(\sigma-\phi(\sigma)/k_2)/d\phi(\sigma) = [d(\sigma-\phi(\sigma)/k_2)/d\sigma][d\sigma/d\phi(\sigma)] =$$

$$[1 - (d\phi(\sigma)/d\sigma)(1/k_2)][d\sigma/d\phi(\sigma)] \geq 0\ , \tag{1.10}$$

showing that $\sigma - \phi(\sigma)/k_2$ is a monotone increasing function of $\phi(\sigma)$. Applying the lemma gives

$$I(T) \leq \int_0^{T_n} \left(\sigma_+^n(t-T) - \frac{\phi_+^n(t-T)}{k_2}\right) \phi_+^n(t)\, dt + \int_0^{T_n} \left(\sigma_-^n(t-T) - \frac{\phi_-^n(t-T)}{k_2}\right) \times$$

$$\phi_-^n(t)\, dt \leq \int_0^{T_n} \left(\sigma_+^n(t) - \frac{\phi_+^n(t)}{k_2}\right) \phi_+^n(t)\, dt + \int_0^{T_n} \left(\sigma_-^n(t) - \frac{\phi_-^n(t)}{k_2}\right) \phi_-^n(t)\, dt$$

$$= \int_0^{T_n} \left(\sigma(t) - \frac{\phi(t)}{k_2}\right) \phi(t)\, dt \,. \tag{1.11}$$

Using (1.6) and (1.7) gives for that part of (1.4) involving $x(t)+y(t)$

$$\int_{-\infty}^{+\infty} (x'(\lambda) + y'(\lambda)) I(\lambda)\, d\lambda + \sum_{i=1}^{u} a_i I(-b_i) + \sum_{j=1}^{v} c_i I(d_i). \tag{1.12}$$

Now, since $x'(\lambda)$, $y'(\lambda)$, a_i and c_i are non-positive, application of (1.11) and (1.12) yields

$$\int_0^{T_n} ((x(t) + y(t)) * (\sigma^n(t) - \phi^n(t)/k_2))\, \phi^n(t)\, dt \geq$$

$$\left[\int_{-\infty}^{+\infty} (x'(\lambda) + y'(\lambda))\, d\lambda + \sum_{i=1}^{u} a_i + \sum_{j=1}^{v} c_i\right] I(0)\,. \tag{1.13}$$

Using (1.3) from the statement of the theorem it follows that the left hand side of (1.13) is greater than $-I(0)$ and hence that the assertion of (1.4) is correct.

The next step in the proof is to apply Parseval's theorem to a part of (1.4) and to use the frequency domain condition (1.1). Let $\sigma^n(t) = \sigma_\phi{}^n(t) + \sigma_i{}^n(t)$ and $\dot\sigma^n(t) = \dot\sigma_\phi{}^n(t) + \dot\sigma_i{}^n(t)$ where $\sigma_\phi{}^n(t)$ and $\dot\sigma_\phi{}^n(t)$ are those components of $\sigma^n(t)$ and $\dot\sigma^n(t)$, respectively, due to the feedback signal $-\phi(t)$ and $\sigma_i^{\,n}(t)$ and $\dot{\sigma}_i^{\,n}(t)$ are due to the initial condition excitation of the system. Then

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * (\sigma^n(t) - \phi^n(t)/k_2))\ \phi^n(t)\ dt +$$

$$\alpha \int_0^{T_n} \dot{\sigma}^n(t)\ \phi^n(t)\ dt = \int_0^{T_n} ((\delta(t) + y(t)) * (\sigma_\phi^{\,n}(t) - \phi^n(t)/k_2))\ \phi^n(t)\ dt +$$

$$\int_0^{T_n} (x(t) * (\sigma_\phi^{\,n}(t) - \phi^n(t)/k_2))\ \phi^n(t)\ dt + \alpha \int_0^{T_n} \dot{\sigma}_\phi^{\,n}(t)\ \phi^n(t)\ dt +$$

$$+ \int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma_i^{\,n}(t))\ \phi^n(t)\ dt + \alpha \int_0^{T_n} \dot{\sigma}_i^{\,n}(t)\ \phi^n(t)\ dt. \tag{1.14}$$

Several substitutions will be made in the integrands on the right hand side of (1.14). In the first and third integrals let $\sigma_\phi^{\,n}(t)$ be replaced by $\sigma_\phi^{\,n\,*}(t)$ and $\dot{\sigma}_\phi^{\,n}(t)$ by $\dot{\sigma}_\phi^{\,n\,*}(t)$ respectively where

$$\sigma_\phi^{\,n\,*}(t) = -\ F^{-1}\ [G(j\omega)\ F[\phi^n(t)]]$$

and

$$\dot{\sigma}_\phi^{\,n\,*}(t) = -F^{-1}\ [j\ \omega\ G(j\omega)\ F[\phi^n(t)]]$$

with F and F^{-1} denoting the direct and inverse Fourier transform operations, respectively. The values of these integrals are unchanged since the starred quantities are equal to their unstarred counterparts in $(0, T_n)$. The value of $\sigma_\phi^{\,n\,*}(t)$ for $t > T_n$ does not affect the first integral since $\delta(t) + y(t) = 0$ for $t < 0$ and $\phi^n(t) = 0$ for $t > T_n$. The latter reason also shows that the third integral is not influenced by the values of $\dot{\sigma}_\phi^{\,n\,*}(t)$ for $t > T_n$. In the case of the second integral

$x(t)$ being non-zero for $t < 0$ implies that $\sigma_\phi{}^n(t)$ cannot be replaced by $\sigma_\phi{}^{n\,*}(t)$ without changing the value of this integral. Therefore, the portion of $\sigma_\phi{}^{n\,*}(t)$ for $t > T_n$ must be taken into account in making the substitution. Let

$$\sigma_\phi{}^{n\,*}(t) = \sigma_\phi{}^n(t) + \sigma_\phi{}^d(t) \qquad (1.15)$$

where $\sigma_\phi{}^d(t)$ is that component of $\sigma_\phi{}^{n\,*}(t)$ occurring in (T_n,∞). With these substitutions the first three integrals on the right hand side of (1.14) are

$$\int_o^{T_n} ((\delta(t) + y(t)) * (\sigma_\phi{}^{n*}(t) - \phi^n(t)/k_2))\ \phi^n(t)\ dt +$$

$$\int_o^{T_n} (x(t) * (\sigma_\phi{}^{n*}(t) - \phi^n(t)/k_2))\ \phi^n(t) dt + \alpha \int_o^{T_n} \dot{\sigma}_\phi{}^{n*}(t)\ \phi^n(t)\ dt$$

$$- \int_o^{T_n} (\sigma_\phi{}^d(t) * x(t))\ \phi^n(t)\ dt\ . \qquad (1.16)$$

For the final step in the proof a bound is required on the last integral of (1.16) in terms of $|\phi^n(t)|$max, the largest value of $|\phi^n(t)|$ in $(0,T_n)$. $|\sigma_\phi{}^d(t)|$ is given by

$$\left| \int_o^t g(\lambda)\ \phi^n(t-\lambda)\ d\lambda \right|\ ,\ t \geq T_n\ , \qquad (1.17)$$

where $g(\lambda) = F^{-1}(G(j\omega))$. Because of condition b of the theorem, it is possible to find two positive numbers q and r such that $|g(\lambda)| < q\exp(-r\lambda)$. Using this bound gives

$$|\sigma_\phi{}^d(t)| \leq \int_{t-T_n}^t q\exp(-r\lambda)\ |\phi^n(t)|_{max}\ d\lambda =$$

$$(q/r)\ |\phi^n(t)|_{max} \exp(-rt)\ [\exp(rT_n)-1],\ t \geq T_n\ . \qquad (1.18)$$

The lower limit on the integral has been changed to $t - T_n$ since $\phi^n(t)$ is zero outside $(0, T_n)$.

The piecewise continuous and impulsive components of $x(t)$ will be considered separately. Since $|x'(t)| < \ell \exp(ft)$, using (1.18) gives

$$|\sigma_\phi{}^d(t) * x'(t)| \leq$$

$$\frac{\ell q}{r} |\phi^n(t)|_{max} \int_{-\infty}^{-T_n+t} \exp(f\lambda) \exp(-r(t-\lambda))[\exp(rT_n)-1]d\lambda$$

$$= \frac{\ell q}{r(r+f)} |\phi^n(t)|_{max} [\exp(rT_n)-1] \exp(-(r+f)T_n) \exp(ft) \qquad (1.19)$$

$$0 \leq t \leq T_n.$$

Using this result gives

$$\int_o^{T_n} (\sigma_\phi{}^d(t) * x'(t)) \phi^n(t)dt \leq |\phi^n(t)|_{max} \int_o^{T_n} |\sigma_\phi{}^d(t) * x'(t)|dt \leq$$

$$\frac{\ell q|\phi^n(t)|^2{}_{max}}{rf(r+f)} (1 - \exp(-rT_n))(1 - \exp(-fT_n)) \leq M_1|\phi^n(t)|^2_{max}$$

(1.20)

where M_1 is a positive number independent of T_n.

For the impulsive case,

$$\sigma_\phi{}^d(t) * x(t) = \sum_{i=1}^{u} a_i \sigma_\phi{}^d(t+b_i) \qquad (1.21)$$

and

$$\left| \int_o^{T_n} (\sigma_\phi{}^d(t) * x(t)) \phi^n(t)dt \right| \leq \sum_{i=1}^{u} |a_i| \int_o^{T_n} |\sigma_\phi{}^d(t+b_i)\phi^n(t)|dt.$$

(1.22)

If $b_i < T_n$, the use of (1.18) in the right hand side integral of (1.22) gives

$$\int_o^{T_n} |\sigma_\phi{}^d(t+b_i)\phi^n(t)dt| \le \frac{q}{r} |\phi^n(t)|^2_{max} [\exp(rT_n)-1] \int_{T_n-b_i}^{T_n} \exp(-r(t+b_i))dt$$

(1.23)

The lower limit on the right hand side integral is T_n-b_i since $\sigma_\phi{}^d(t+b_i) = 0$ for $t < b_i - T_n$. Evaluating (1.23) gives

$$\frac{q}{r^2} |\phi^n(t)|^2_{max} [1-\exp(-rT_n)][1-\exp(-rb_i)] \le M_{2i}|\phi^n(t)|^2_{max} \qquad (1.24)$$

where M_{2i} is a positive number. Finally, if $b_i > T_n$, the left hand side of (1.23) is less than or equal to

$$\frac{q}{r} |\phi^n(t)|^2_{max}[\exp(rT_n)-1] \int_o^{T_n} \exp(-r(t+b_i))dt = \frac{q}{r^2}|\phi^n(t)|^2_{max} \times$$

$$[\exp(-r(b_i - T_n)) - \exp(-r\, b_i)][1 - \exp(-r\, T_n)] \le M_{3i}|\phi^n(t)|^2_{max}$$

(1.25)

where M_{3i} is a positive number. Using (1.20), (1.24), and (1.25) gives

$$|\int_o^{T_n} (\sigma_\phi{}^d(t) * x(t))\ \phi^n(t)dt| \le (M_1 + \sum_{i=1}^{u} |a_i|M_3)\ |\phi^n(t)|^2_{max}$$

$$= M|\phi^n(t)|^2_{max} \qquad (1.26)$$

where M_3 is the largest of the M_{2i}'s and M_{3i}'s and M is a positive number independent of T_n. That is the desired bound.

Since $\phi^n(t)$ is zero outside $(0,T_n)$, the limits on the first 3 integrals of (1.16) may be changed to $(-\infty,\infty)$. Also, because of the conditions on the various functions involved, Parseval's Theorem is applicable to these integrals. Its application gives

$$\int_{-\infty}^{+\infty} ((\delta(t) + x(t) + y(t)) * (\sigma_\phi^{n*}(t) - \phi^n(t)/k_2))\, \phi^n(t)\, dt +$$

$$\alpha \int_{-\infty}^{+\infty} \dot{\sigma}_\phi^{n*}(t)\, \phi^n(t) dt = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} ((1+X(j\omega) + Y(j\omega))(G(j\omega) + 1/k_2)$$

$$+ \alpha j\, \omega\, G(j\omega)) \quad |F[\phi^n(t)]|^2 d\omega \qquad (1.27)$$

Since the imaginary part of the integral on the right hand side of (1.27) is zero, (1.27) may be rewritten as

$$-\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{Re}(1 + X(j\omega) + Y(j\omega) + \alpha j\omega)(G(j\omega) + 1/k_2) \; |F[\phi^n(t)]|^2 d\omega. \qquad (1.28)$$

From (1.1) it follows that (1.28) is non-positive. Combining (1.4), (1.14), (1.16), (1.26), (1.27), and (1.28) gives

$$c(T_n) \int_0^{T_n} (\sigma^n(t) - \phi^n(t)/k_2)\dot{\phi}^n(t) dt + \alpha\Phi(T_n) - \alpha\Phi(0) \le$$

$$\left|\int_0^{T_n} (\sigma_\phi^d(t) * x(t))\, \phi^n(t) dt\right| + \left|\int_0^{T_n} (\sigma_i^n(t) + (x(t) * \sigma_i^n(t)) +\right.$$

$$\left.(y(t) * \sigma_i^n(t)) + \alpha\dot{\sigma}_i^n(t))\, \phi^n(t)\, dt\right| \qquad (1.29)$$

$$\le M|\phi^n(t)|^2_{max} + P|\phi^n(t)|_{max} \qquad (1.30)$$

where

$$P = \int_0^{\infty} |\sigma_i(t) + x(t) * \sigma_i(t) + y(t) * \sigma_i(t) + \alpha\dot{\sigma}_i(t)| dt$$

and

$$\Phi(T_n) = \int_o^{\sigma(T_n)} \phi(\sigma)\, d\sigma.$$

Therefore,

$$\Phi(T_n) \leq \frac{1}{\alpha} [M \, |\phi^n(t)|^2_{max} + P|\phi^n(t)|_{max}] + \Phi(0). \qquad (1.31)$$

Using the approach given in Lefschetz [11], let T_n be chosen such that $|\phi^n(t)|_{max}$ occurs at T_n. Then with the first part of condition c holding, it follows that σ and hence $\phi(\sigma)$ are bounded; if this were not the case, inequality (1.31) would not hold for large values of $|\sigma|$. If the second part of condition c holds, a quadratic Liapunov function may be found using the approach of Rekasius [12] that shows the boundedness of σ and $\phi(\sigma)$.

Since the right hand side of (1.31) is bounded, it follows from (1.30) that $\int_o^{T_n} (\sigma^n(t) - \phi^n(t)/k_2)\, \phi^n(t)dt$ is bounded, from which asymptotic stability in the large follows, using the arguments given in Aizerman and Gantmacher [13]. This completes the proof of the theorem.

In order to prove corollary 1, the lemma is applied directly to (1.8) to give $|I(T)| \leq I(0)$ instead of $I(T) \leq I(0)$. (1.13) then becomes

$$\left| \int_o^{T_n} ((x(t) + y(t)) * (\sigma^n(t) - \phi^n(t)/k_2))\, \phi^n(t)dt \right| \leq$$

$$[\int_{-\infty}^{+\infty} (|x'(\lambda)| + |y'(\lambda)|)d\lambda + \sum_{i=1}^{u} |a_i| + \sum_{j=1}^{v} |c_i|]\ I(0). \qquad (1.32)$$

Using the condition of this corollary, it follows that the left hand side of (1.32) is less than or equal to I(0), from which (1.4) follows. The remainder of the proof is unchanged. This completes the proof of corollary 1.

To prove the assertion of corollary 2, it is first shown that if

$$\text{Re } Z(G + 1/k_2) \geq \delta_2 > 0,$$

$$\text{Re } Z(G/(1 + \varepsilon G) + 1/k_2) \geq \delta_3 > 0$$

for ε sufficiently small. δ_3 is a positive number. By a straightforward calculation $\text{Re } Z(G/(1 + \varepsilon G) + 1/k_2)$ is

$$\frac{\text{Re } Z\ (G + 1/k_2) + \varepsilon(\text{Re } Z\ )\ [|G|^2\ (1 + \varepsilon/k_2) + 2(\text{Re } G)/k_2]}{(1 + \varepsilon R)^2 + (\varepsilon X)^2}$$

The first quantity in the numerator is non-negative. Since Re Z is non-negative, the second quantity in the numerator may be negative if $-2/(k_2 + \varepsilon) < \text{Re } G < 0$. For this interval ε must be chosen small enough such that the numerator is positive. This is guaranteed by having

$$\varepsilon < \frac{-\delta_2 k_2}{2\text{Re } Z \quad \text{Re } G}$$

in the interval. Let the linear transformation $\phi_1(\sigma) = \phi(\sigma) - \varepsilon\sigma$ be applied to the system. Then $G_1 = G/(1 + \varepsilon\, G)$. The stability of the transformed system will guarantee the stability of the original system. If ε is chosen to be less than both δ and the right hand side of the ε inequality, the transformed system will satisfy the conditions of the theorem for the noncritical cases. Q.E.D.

The proof of corollary 3 follows directly from the proof of the theorem with $x(t) = 0$. (1.31) becomes $\Phi(T_n) \leq \frac{P}{\alpha} |\phi^n(t)|_{max} + \Phi(0)$. Since $\phi(\sigma)$ is a monotone increasing function of σ, for $|\sigma|$ sufficiently large the left hand side of this inequality will become greater than the right, showing that $\sigma(t)$ and $\phi(\sigma(t))$ are bounded. The remainder of the proof is unchanged.

F. Theorem for a Nonlinearity With a Monotone Bound

This theorem is an improved version of one given in [4]. The two improvements consist of permitting Z(s) to have a corresponding time function that is non-zero for $t < 0$ and of taking the symmetry of the nonlinearity into account, resulting in $x(t)$ and $y(t)$ being allowed to take on positive as well as negative values.

Theorem 1.2. For the system given in figure 1 let the following conditions hold:

a. $A\phi_m(\sigma)\,\sigma \leq \phi(\sigma)\,\sigma \leq B\phi_m(\sigma)\,\sigma$, where A and B are real numbers satisfying $0 < A \leq 1$ and $1 \leq B < \infty$, $\phi(0) = \phi_m(0) = 0$, $\sigma\,\phi(\sigma) < k\,\sigma^2$ where $k > 0$ and $\sigma\,\phi_m(\sigma) > 0$ for $\sigma \neq 0$, $d\phi(\sigma)/d\sigma$ is a continuous function of σ, $\phi_m(\sigma)$ is a continuous monotone increasing function of σ having an odd part $\phi_{mo}(\sigma)$ that satisfies $|\phi_m(\sigma)| \leq C|\phi_{mo}(\sigma)|$ and $|\phi_{mo}(\sigma)| \leq D|\phi_m(\sigma)|$.

b. Conditions b and c of theorem 1.1.

Then a sufficient condition for asymptotic stability in the large is that

$$\mathrm{Re}\ [Z(j\omega)\ G(j\omega) + E\ (G(j\omega) + 1/k)] \geq 0 \tag{1.33}$$

for all real ω where E is a non-negative number. $Z(j\omega)$ is defined as in (1.2) but (1.3) becomes

$$\frac{BCD}{A}\ [\int_{-\infty}^{+\infty} (x'^{+}(t) + y'^{+}(t))\ dt + \sum a_i^{+} + \sum c_i^{+}] -$$

$$\frac{B}{A}\ [\int_{-\infty}^{+\infty} (x'^{-}(t) + y'^{-}(t))dt + \sum a_i^{-} + \sum c_i^{-}] < 1 \tag{1.34}$$

where $x'^{+}(t)$, y'^{+}, a_i^{+}, and c_i^{+} are the positive portions or values of the corresponding non-superscripted functions or numbers and $x'^{-}(t)$, $y'^{-}(t)$, a_i^{-}, and c_i^{-} are the negative portions or values of the corresponding non-superscripted functions or numbers.

<u>Proof</u>. Starting with (1.4) of the proof of theorem 1.1, let this equation be replaced by

$$\int_{o}^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt = c(T_n) \int_{o}^{T_n} \sigma^n(t)\phi^n(t)dt \tag{1.35}$$

as the condition to be shown. Repeating the steps used to obtain (1.6) and (1.7) gives

$$\int_{-\infty}^{o} (x'^{+}(\lambda) + x'^{-}(\lambda)) \int_{o}^{T_n} \sigma(t - \lambda)\ \phi^n(t)dt\ d\lambda +$$

$$\int_{o}^{\infty} (y'^{+}(\lambda) + y'^{-}(\lambda)) \int_{o}^{T_n} \sigma(t - \lambda)\ \phi^n(t)\ dt\ d\lambda \tag{1.36}$$

and

$$\sum a_i^+ \int_o^{T_n} \sigma^n(t + b_i)\ \phi^n(t)dt + \sum a_i^- \int_o^{T_n} \sigma^n(t + b_i)\phi^n(t)\ dt$$

$$+ \sum c_i^+ \int_o^{T_n} \sigma^n(t-d_i)\ \phi^n(t)dt + \sum c_i^- \int_o^{T_n} \sigma^n(t-d_i)\phi^n(t)dt. \quad (1.37)$$

$I(T)$ then becomes $I(T) = \int_o^{T_n} \sigma^n(t - T)\ \phi^n(t)dt$. At this point the proof differs from that of theorem 1 for it is desired to develop both positive and negative bounds on $I(T)$. First a bound is developed on $|I(T)|$.

$$|I(T)| \leq B \int_o^{T_n} |\sigma^n(t - T)\ \phi_m^{\ n}(t)|dt$$

$$\leq BC \int_o^{T_n} |\sigma^n(t - T)\ \phi_{mo}^{\ n}(t)|dt \leq BC \int_o^{T_n} \sigma^n(t)\ \phi_{mo}^{\ n}(t)dt \quad (1.38)$$

where use has been made of the lemma. $\phi_m^{\ n}(t) = \phi_m^{\ n}(\sigma(t))$ and $\phi_{mo}^{\ n}(t) = \phi_{mo}^{\ n}(\sigma(t))$. Continuing the development gives

$$BC \int_o^{T_n} \sigma^n(t)\ \phi_{mo}^{\ n}(t)dt \leq BCD \int_o^{T_n} \sigma^{\ n}(t)\ \phi_m^{\ n}(t)$$

$$\leq \frac{BCD}{A}\int_o^{T_n} \sigma^n(t)\ \phi^n(t)dt\ . \quad (1.39)$$

The negative bound on $I(T)$ is then

$$I(T) \geq -\frac{BCD}{A}\ I(0)\ . \quad (1.40)$$

For the positive bound the same procedure as in theorem 1 is used to give

$$\int_0^{T_n} \sigma^n(t-T)\phi^n(t)dt \le \int_0^{T_n} \sigma_+^n(t-T)\ \phi_+^n(t)dt + \int_0^{T_n} \sigma_-^n(t-T)\phi_-^n(t)dt$$

$$\le B \int_0^{T_n} \sigma_+^n(t-T)\ \phi_{m+}{}^n(t)dt + B\int_0^{T_n} \sigma_-^n(t-T)\ \phi_{m-}{}^n(t)dt$$

$$\le B\int_0^{T_n} \sigma^n(t)\ \phi_m{}^n(t)dt \le \frac{B}{A}\int_0^{T_n} \sigma^n(t)\ \phi^n(t)dt\ . \tag{1.41}$$

Using these two bounds in (1.36) and (1.37) gives

$$\int_0^{T_n} ((x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt \ge$$

$$-\frac{BCD}{A}\ [\int_{-\infty}^{+\infty} (x'^+(\lambda) + y'^+(\lambda))\ d\lambda + \sum a_i{}^+ + \sum c_i{}^+]\ I(0)$$

$$+\frac{B}{A}\ [\int_{-\infty}^{+\infty} (x'^-(\lambda)+y'^-(\lambda))d\lambda + \sum a_i{}^- + \sum c_i{}^-]\ I(0)\ . \tag{1.42}$$

Using (1.42) and (1.34) gives (1.35). The remainder of the proof is similar to that of theorem 1 with the left hand side of (1.14) replaced by

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt + \alpha\int_0^{T_n} \dot\sigma^n(t)\ \phi^n(t)dt$$

$$+ E\int_0^{T_n} (\sigma^n(t) - \phi^n(t)/k)\ \phi^n(t)dt\ . \tag{1.43}$$

Q.E.D.

The frequency domain condition (1.33) is certainly not as easy to apply as (1.1). (1.33) was obtained because of the necessity of using (1.35) in order to apply the various conditions on $\phi(\sigma)$. An example of the application of this theorem is considered next.

Example 3. Let $\phi(\sigma)$ be an odd function defined for positive values of σ by

$$\begin{aligned} \phi(\sigma) &= \sigma &&, \ 0 \le \sigma \le 1.25 \\ &= -\sigma + 2.5 &&, \ 1.25 \le \sigma \le 1.5 \\ &= (5\sigma/3)/(1+\sigma) &&, \ 1.50 \le \sigma \end{aligned}$$

and let $G(s) = K(s + 4)(s + 50)^2/(s + .1)(s + 1)(s + 1000)^2$, with K being large but finite. It is assumed that the kinks in the $\phi(\sigma)$ curve are smoothed out so that the derivative is a continuous function of σ. A plot of this nonlinear characteristic reveals that a convenient choice is to take $\phi_m(\sigma)$ as an odd function equal to $\phi(\sigma)$ for positive values of σ except for $1.25 \le \sigma \le 3.01$ for which interval $\phi_m(\sigma) = 1.25$. $\phi_m(\sigma)$ is then a continuous odd monotone increasing function of σ. With this choice A = .8, B = C = D = 1 and (1.34) becomes $\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)dt < .8$. Since K is to be large but finite, let E = 0 to give Re $Z(j\omega)\, G(j\omega) \ge 0$ as the criterion to be satisfied. $G(j\omega)$ has an angle that lies outside the $\pm 90^{\circ}$ band in a lagging direction at low frequencies, at higher frequencies the angle approaches $+ 90^{\circ}$ and then $- 90^{\circ}$ at very high frequencies. Because of this behavior, the Popov criterion will not show stability. Let $Z(s) = (s + 1)(s + 1000)/(s + 4)$. This particular function has the proper phase characteristic, that is, leading at low frequencies, almost zero at intermediate frequencies, and then leading at high frequencies to give a product with an angle in the $\pm 90^\circ$ band. Since $Z(s)\ G(s) = K\ (s + 50)^2/(s + .1)(s + 1000)$, it is seen that Re $Z(j\omega)\ G(j\omega) > 0$ for all ω. Expressing $Z(s)$ in a partial fraction expansion form gives $Z(s) = s + 997 - 2988/(s + 4)$. The left hand side of (I.34) is .937, and hence this condition is satisfied. Therefore, the given system is asymptotically stable in the large.

G. Conclusion

This chapter has presented two theorems which allow the $Z(s)$ multiplier to correspond to a function of time that is non-zero for $t < 0$ as well as for $t > 0$. This innovation solves the problem of obtaining a $Z(j\omega)$ whose angle varies with equal freedom between $0^\circ - +90^\circ$ and $0^\circ - -90^\circ$. The generalized RL $Z(s)$ multiplier considered shows that a nonlinear system having a monotone non-linearity with a slope in the sector $(0,k_2)$ is stable provided that the system is stable for linear gains in the sector $(0,k_2)$ and provided that the angle changes slowly enough with frequency. Although this work gives improved results, it is not clear how close these results are to the actual absolute stability limit. Additional study is needed to resolve this matter.

While the two Z(s) functions discussed appear to be quite useful, if it is not possible to show stability with either of these two, it is not clear how one should go about generating additional Z(s) functions with more desirable characteristics, other than to use trial and error. The reason for this is the need to consider simultaneously both the time and the frequency domain behavior of a possible candidate for a Z(s) function. This appears to be a worthwhile area for further research.

Condition c of theorem 1.1 is one way of guaranteeing the boundedness of $\sigma(t)$ and $\phi(t)$. If a certain nonlinearity does not satisfy this condition, the theorem may still be applied provided that a Liapunov function can be found that will show the boundedness of the state variables of the system. However, finding a suitable Liapunov function may be a difficult task.

II. Appendix 1

Lemma. If $f_a(t)$ and $f_b(t)$ are two continuous time functions which are zero outside the time interval $(0, T_n)$, $f_b(t) = h(f_a(t))$ where h is a piecewise continuous monotone increasing function of f_a, and if either $f_a(t)$ and $f_b(t)$ are both always non-negative or non-positive or h is an odd monotone function with $h(0) = 0$, then

$$\int_o^{T_n} (f_a(t)\ f_b(t) - |f_a(t)\ f_b(t + T)|)dt \geq 0$$

for any real value of T.

Proof. Given a value of $T > 0$, let the summation

$$\sum_{i=1}^{n} |f_a(\delta i)\ f_b(\delta i + T)|\ \delta \qquad \text{(A1)}$$

be formed where δ is a positive number chosen such that T/δ is an integer and n is chosen such that $n\delta = T_n - \delta_1$ where δ_1 is a positive number less than δ. Let a ranking of the magnitudes of the values of $f_a(t)$ and $f_b(t)$ that can appear in the summation be set up such that $|f_{a1}| \geq |f_{a2}| \geq |f_{a3}|\ldots$ for f_a and a similar ordering $|f_{b1}| \geq |f_{b2}| \geq |f_{b3}|\ldots$ holds for f_b. Since h is monotone increasing and either an odd function or $f_a(t)$ and $f_b(t)$ are both always non-positive or non-negative, values of $|f_{ai}|$ and $|f_{bj}|$ with the same numerical subscript occur at the same time or the ranking can be arranged such that they occur at the same time if two or more magnitudes are equal. Using the ranked magnitudes, a table of product values that may appear in the summation is formed as indicated below.

$$
\begin{array}{cccccccc}
 & |f_{b1}| & |f_{b2}| & |f_{b3}| & \cdots & |f_{bj}| & \cdots & |f_{bn}| \\
|f_{a1}| & f_{a1}f_{b1} & |f_{a1}f_{b2}| & & & & & \\
|f_{a2}| & |f_{a2}f_{b1}| & f_{a2}f_{b2} & & & & & \\
|f_{a3}| & & & & & & & \\
\vdots & & & & & & & \\
|f_{ai}| & & & & & & & \\
\vdots & & & & & & & \\
|f_{an}| & & & & & & & f_{an}f_{bn}
\end{array}
$$

The diagonal elements in this table correspond to the terms that appear in (A1) with $T = 0$. For any value of T, the terms $|f_{ai}|$ and $|f_{bj}|$ can appear only once, if at all, in the summation. This means that of the product elements appearing in (A1), only one element can occur in a given row and one element in a given column in the table of product values. Also, for $T \neq 0$, the summation terms appear as off diagonal elements in the table. Next, by using a row and column counting process it will be shown that

$$\sum_{i=1}^{n} f_a(\delta i)\ f_b(\delta i) \geq \sum_{i=1}^{n} |f_a(\delta i)\ f_b(\delta i + T)| \qquad \text{(A2)}$$

for $T \neq 0$.

Consider the elements on the right hand side of (A2) that appear in the first row or first column of the table of product values. The maximum possible number is two. If it is zero or one, an inequality $f_{a1}f_{b1} \geq 0$, $f_{a1}f_{b1} \geq |f_{a1}f_{bj}|$ or $f_{a1}f_{b1} \geq |f_{ai}f_{b1}|$ is formed. The first row and the first column are then removed, giving a reduced table of product values. If there are two elements, it it necessary to consider three cases.

a. The two terms are $|f_{aj}f_{b1}|$ and $|f_{a1}f_{bj}|$. In this case the two diagonal terms $f_{a1}f_{b1}$ and $f_{aj}f_{bj}$ are used to give the inequality $f_{a1}f_{b1} + f_{aj}f_{bj} \geq |f_{aj}f_{b1}| + |f_{a1}f_{bj}|$. Since the only two elements possible in the first and jth rows and columns have been bounded by the diagonal terms associated with these rows and columns, the first and jth rows and columns are removed, giving a reduced table of product values.

b. The two terms are $|f_{ai}f_{b1}|$ and $|f_{a1}f_{bj}|$ with $i < j$. An inequality that may be written is $f_{a1}f_{b1} + f_{ai}f_{bi} \geq |f_{ai}f_{b1}| + |f_{a1}f_{bi}|$. If there is no term in the ith column, $|f_{a1}f_{bi}|$ is used to bound $|f_{a1}f_{bj}|$, since $|f_{a1}f_{bi}| \geq |f_{a1}f_{bj}|$, giving as the desired inequality $f_{a1}f_{b1} + f_{ai}f_{bi} \geq |f_{ai}f_{b1}| + |f_{a1}f_{bj}|$. The first and ith rows and columns are then removed to give a reduced table of product values. If there is a term in the ith column, say $|f_{ak}f_{bi}|$, the $|f_{ak}f_{bi}|$ and $|f_{a1}f_{bj}|$ terms are bounded by the $|f_{a1}f_{bi}|$ term and the $|f_{ak}f_{bj}|$

term, giving the inequality $|f_{a1}f_{bi}| + |f_{ak}f_{bj}| \geq |f_{a1}f_{bj}| + |f_{ak}f_{bi}|$. Combining this bound with the one involving $|f_{ai}f_{b1}|$ gives $f_{a1}f_{b1} + f_{ai}f_{bi} + |f_{ak}f_{bj}| \geq |f_{ai}f_{b1}| + |f_{a1}f_{bj}| + |f_{ak}f_{bi}|$ as the overall inequality resulting from this step. The $|f_{ak}f_{bj}|$ term has been borrowed to obtain the bound. This term is not an element of the summation since the kth row and jth columns by hypothesis each have one element. A reduced table of product values is obtained by deleting the first and ith rows and columns and adding the $|f_{ak}f_{bj}|$ term as one to be bounded by the remaining diagonal elements. The array obtained has the same properties as the original array with regard to each row and column having only one element. Therefore, the process may be repeated on the reduced product value table.

c. The two terms are $|f_{ai}f_{b1}|$ and $|f_{a1}f_{bj}|$ with $i > j$. The strategy of b is repeated with the roles of the ith and jth column being taken by the jth and ith rows, respectively. The process is then applied to the first row and column of the reduced table of product values and repeated until there are no terms left in the final reduced table. Adding together the inequalities obtained at each stage of the process gives the left hand side of (A2) plus additional terms greater than the right hand side of (A2) plus the same additional terms. Upon cancelling the common terms, (A2) results. From (A2) it follows that

$$\delta \sum_{i=1}^{n} \left(f_a(\delta i)\, f_b(\delta i) - |f_a(\delta i)\, f_b(\delta i + T)|\right) \geq 0.$$

Since

$$\int_{o}^{T_n} (f_a(t)f_b(t) - |f_a(t)f_b(t + T)|)dt =$$

$$\sum_{i=1}^{n} (f_a(\delta i)\ f_b(\delta i) - |f_a(\delta i)f_b(\delta i + T)|)\delta + F,$$

where F is a real number that can be made arbitrarily small by a suitable choice of δ, taking the limit as $\delta \to o$ gives the assertion of the lemma for positive T. A similar discussion shows that the lemma also holds for negative T. Q.E.D.

IV CHAPTER II. BOUNDS ON THE RESPONSE OF AN AUTONOMOUS SYSTEM WITH A SINGLE NONLINEARITY

A. Introduction

This chapter is concerned with the calculation of bounds on the response of the single nonlinearity system of Figure 1. For the first theorems it is assumed that the external input to the system is zero and that the system is excited by initial conditions only. Then, Fourier transformable inputs of a certain class are permitted in later theorems. If the input is itself bounded, the bounds which are calculated on the response enable the showing of Liapunov stability but not asymptotic stability. The bound that is determined is on the function $\Phi(\sigma(t))$ and usually takes one of the forms shown in figure 6. Once a bound has been obtained on $\Phi(\sigma(t))$, a bound can be calculated for $\sigma(t)$ for specific nonlinear characteristics.

Pertinent references include the survey paper by Kalman and Bertram [14] in which it is pointed out that an exponential bound can be obtained on the response by the use of Liapunov functions. The maximum value of $\dot{v}/v = -\eta$ is calculated over the space in which the response is confined. The bound is then $v(t) \leq v(0)\, e^{-\eta t}$. The bound on $v(t)$ can then be converted into a bound on the system variables. Sandberg [15] considered the problem of a time varying nonlinearity confined to a linear sector and gave a frequency domain condition

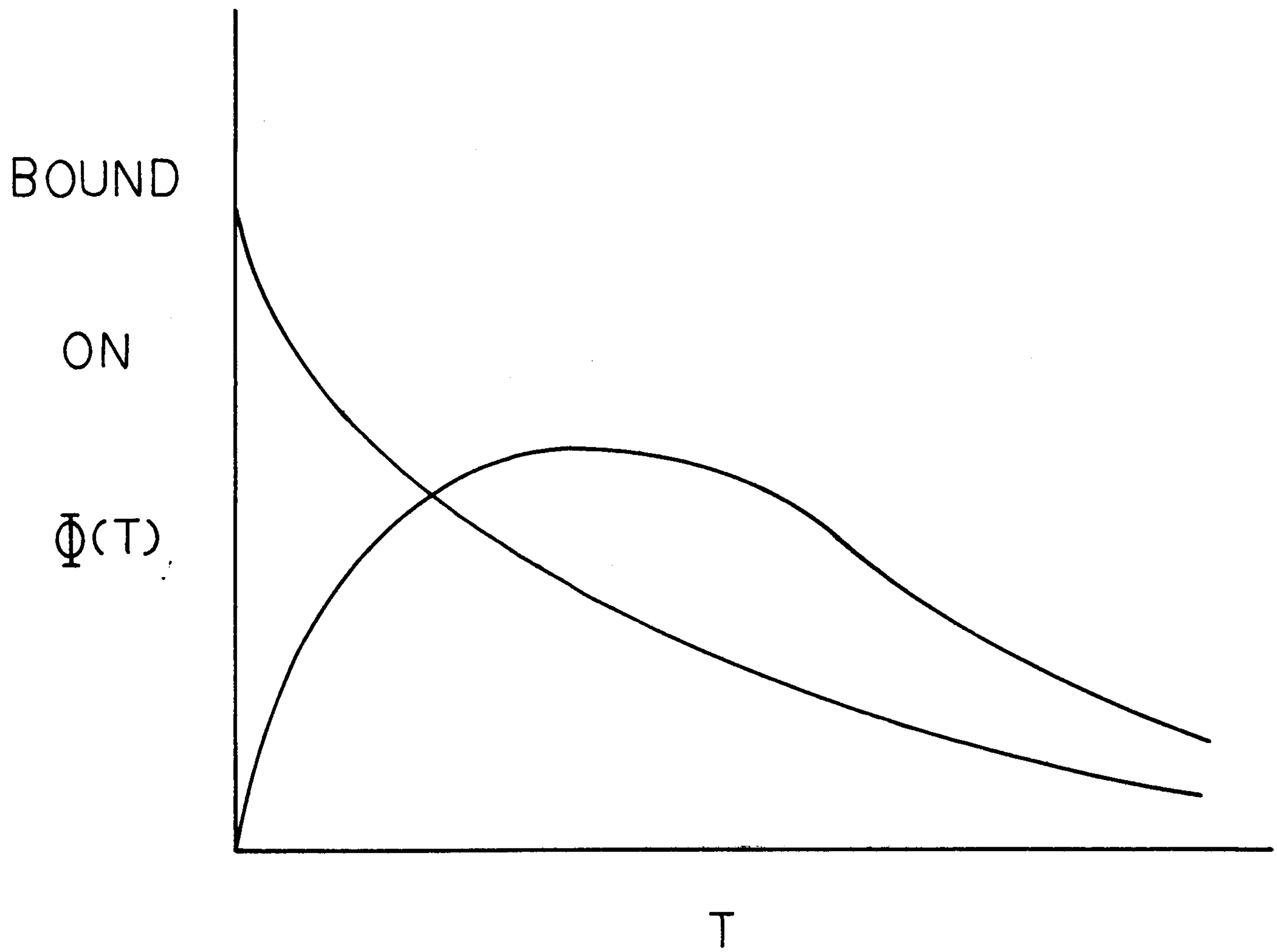

Figure 6. Typical Bounds on $\Phi(\sigma(t))$.

guaranteeing that the state variables approach zero exponentially with time. In the single stationary nonlinearity case with a zero lower bound on the nonlinearity, this frequency domain condition is equivalent to Re $G(j\omega) \geq 0$, which is a rather restricted criterion. Tsypkin [16] obtained an analogous result using a Popov type approach for a sampled data system having a single nonlinearity. Using a Liapunov approach, Yakubovich [17] showed that for a nonlinearity confined to a sector $(0, k)$, if Re $G(j\omega - a)(1 + \alpha j\omega) + 1/k \geq 0$, then the response of the system satisfies $|\sigma(t)| \leq Me^{-at}|\sigma(0)|$ where M is a positive number. This last result is similar to the Popov criterion except for the shift in the argument of $G(j\omega)$.

Although the criteria of the last 3 references show the existence of a bound of the desired type, these references do not consider the problem of calculating a value of M. Also, the corresponding frequency domain stability criteria for these works are more restricted than those given in chapter 1. Therefore, the main object of this paper is to develop theorems giving bounds on the response of systems using the approach employed in the development of the stability criteria of chapter 1. Once a system has been shown to be asymptotically stable in the large using these criteria, it will then be possible to calculate a bound on the response using the results of this chapter.

The first three theorems deal with those systems in which it is possible to show stability with $x(t) = 0$. Theorems 2.4 and 2.5 give bounding expressions for those cases in which $x(t) \neq 0$. Since for this case a bound must be available on the response of the system of the form $\Phi(t) \leq M|\phi(t)|_{max}$, where M is a positive number and $|\phi(t)|_{max}$ is the largest value of $\phi(t)$ in $(0, T_n)$, the application of these latter two theorems requires somewhat more computation than the first 3. The bounds for these first five theorems are calculated using a "completing the square" approach of Aizerman and Gantmacher [13]. Under certain circumstances an improved bound can be found using the approach of Lefschetz [11]. This is used in theorem 2.6 and 2.7. Theorem 2.8 gives a bound on the response with an external input applied and theorem 2.9 considers a special case which arises when dealing with systems having lag compensators. Finally, the possibility of obtaining an improved bound when the system is in the linear region is discussed.

B. The Theorems

Theorem 2.1. For the system of figure 1 excited by initial conditions only let the following hold:

a. $0 \leq d\phi(\sigma)/d\sigma \leq k_2$ where k_2 is a positive number, $\phi(\sigma)$ and $(\sigma - \phi(\sigma)/k_2) = 0$ only for $\sigma = \phi(\sigma) = 0$, and $d\phi(\sigma)/d\sigma$ be a continuous function of σ.

b. $G(s) = N(s)/D(s)$ with the degree of $N(s)$ at least one less than the degree of $D(s)$ and with the zeros of $D(s)$ having negative real parts whose magnitude is greater than or equal to the positive number a.

c. $$\begin{aligned} \text{Re } H(j\omega) = \text{Re}[&c(1 + Y(j\omega))(G(j\omega - a) + 1/k_2) + dj\omega\, G(j\omega - a) \\ &+ adG(j\omega - a)] \geq b > 0 \end{aligned}$$

where b, c, and d are positive numbers, $y(t)$ is composed of delayed impulses and a piecewise continuous function that satisfies $y(t) \leq 0$ for $t > 0$, $y(t) = 0$ for $t < 0$ and

$$\int_0^\infty |y(t)|\, e^{at} dt < 1 . \tag{2.1}$$

Then

$$\Phi(T_n) = \int_0^{\sigma(T_n)} \phi(\sigma)d\sigma \leq e^{-2aT_n} \left[\frac{\int_0^\infty m^2(t)dt}{4d} + \int_0^{\sigma(0)} \phi(\sigma)d\sigma\right] \tag{2.2}$$

where $m(t) = F^{-1}\,[P(j\omega)\; Q(j\omega)]$ with

$$p(t) = e^{at}[(c + 2ad)\, \sigma_i{}^n(t) + d\, \dot{\sigma}_i{}^n(t)] + c(\sigma_i{}^n(t)\; e^{at} * y(t))^n$$

and $Q(j\omega)$ is defined by $1/\text{Re } H(j\omega) = Q(j\omega)\; Q(-j\omega)$. $\sigma_i{}^n(t)$ is equal to the initial condition component of $\sigma(t)$, $\sigma_i(t)$, in $(0, T_n)$ and zero outside this interval. Similarly, $\dot{\sigma}_i{}^n(t)$ is equal to the initial condition component of $\dot{\sigma}_i(t)$ in $(0, T_n)$ and zero elsewhere. $(\sigma_i{}^n(t) * y(t))^n$ is equal to $\sigma_i{}^n(t)$ convolved with $y(t)$ in $(0, T_n)$ and zero elsewhere.

Proof. First it is desired to establish the non-negativeness of certain integrals which play a prominent role in the development. Using integration by parts with $\dot{\sigma}(t)\,\phi(t)$ being integrated gives

$$\int_0^{T_n} e^{2at}\,\dot{\sigma}(t)\,\phi(t)dt = e^{2aT_n}\,\Phi(T_n) - \Phi(0)$$

$$- 2a\int_0^{T_n} e^{2at}\,\Phi(t)dt\ . \tag{2.3}$$

Also,

$$2a\int_0^{T_n} e^{2at}\,\sigma(t)\phi(t)dt - 2a\int_0^{T_n} e^{2at}\,\Phi(t)dt \geq 0 \tag{2.4}$$

since $\sigma(t)\phi(t)$ and $\Phi(t)$ are both non-negative and $\Phi(t) = \int_0^{\sigma(t)} \phi(\sigma)d\sigma \leq \sigma(t)\phi(t)$ because of the monotone increasing property of $\phi(\sigma)$. Adding the first integral of (2.4) to both sides of (2.3) and rearranging gives

$$e^{2aT_n}\,\Phi(T_n) + 2a\int_0^{T_n} e^{2at}\,\sigma(t)\phi(t)dt - 2a\int_0^{T_n} e^{2at}\Phi(t)dt$$

$$= \int_0^{T_n} e^{2at}\,\dot{\sigma}(t)\phi(t)dt + 2a\int_0^{T_n} e^{2at}\,\sigma(t)\phi(t)dt + \Phi(0) \tag{2.5}$$

where the sum of the second and third terms on the left hand side of (2.5) are non-negative by (2.4).

The second relationship to be established is

$$\int_0^{T_n} e^{2at} (\sigma^n(t) - \phi^n(t)/k_2)\, \phi^n(t)\, dt\ +$$

$$\int_0^{T_n} e^{at} (y(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at}))\, \phi^n(t)dt \geq 0\ . \qquad (2.6)$$

Let the impulsive component of $y(t)$ be given by $\sum_{j=1}^{v} c_j\, \delta(t - d_j)$ where the c_j's are negative numbers and the d_j's positive numbers. Substituting this component into the second integral on the left of (2.6) and inserting an e^{-ad_j} inside the integral and e^{+ad_j} outside gives

$$\sum_{j=1}^{v} c_j\, e^{ad_j} \int_0^{T_n} e^{2a(t-d_j)} (\sigma^n(t - d_j) - \phi^n(t-d_j)/k_2)\phi^n(t)dt. \qquad (2.7)$$

With the piecewise continuous component of $y(t)$, $y'(t)$, substituted into the same integral, the result is

$$\int_0^{T_n} e^{at}\, \phi^n(t) \int_0^{\infty} y'(\lambda)(\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\, e^{a(t-\lambda)} d\lambda\ dt\ . \qquad (2.8)$$

Interchanging the order of integration and inserting an $e^{-a\lambda}$ inside the integration with respect to t and $e^{+a\lambda}$ outside gives

$$\int_0^{\infty} y'(\lambda)\, e^{a\lambda} \int_0^{T_n} e^{2a(t-\lambda)} (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\, \phi^n(t)dt\ d\lambda. \qquad (2.9)$$

Appearing in both (2.7) and (2.9) is an integral of the form $\int_0^{T_n} e^{2a(t-\lambda)}(\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\, \phi^n(t)dt$ where $\lambda > 0$. This integral may be rewritten as

$$\int_o^{T_n} e^{2a(t-\lambda)} \left(\sigma_+^n(t-\lambda) - \phi_+^n(t-\lambda)/k_2\right) \phi_+^n(t)dt \; +$$

$$\int_o^{T_n} e^{2a(t-\lambda)} \left(\sigma_-^n(t-\lambda) - \phi_-^n(t-\lambda)/k_2\right) \phi_-^n(t)dt \; +$$

$$\int_o^{T_n} e^{2a(t-\lambda)} \left(\sigma_-^n(t-\lambda) - \phi_-^n(t-\lambda)/k_2\right) \phi_+^n(t)dt \; +$$

$$\int_o^{T_n} e^{2a(t-\lambda)} \left(\sigma_+^n(t-\lambda) - \phi_+^n(t-\lambda)/k_2\right) \phi_-^n(t)dt \; . \tag{2.10}$$

The plus subscript indicates that the function possessing it is equal to the non-subscripted function when the non-subscripted function is positive and zero otherwise. An analogous definition applies to the use of the negative subscript. For example, $\phi_-^n(t) = \phi^n(t)$ for $\phi^n(t) < 0$ and $\phi_-^n(t) = 0$ for $\phi^n(t) \geq 0$. (2.10) is certainly less than or equal to the first two integrals of this equation. Applying lemma 2 given in the appendix of this chapter to these two integrals gives that (2.10) is less than or equal to

$$\int_o^{T_n} e^{2at}\left(\sigma_+^n(t) - \phi_+^n(t)/k_2\right) \phi_+^n(t)\, dt \; +$$

$$\int_o^{T_n} e^{2at}\left(\sigma_-^n(t) - \phi_-^n(t)/k_2\right) \phi_-^n(t)dt = \int_o^{T_n} e^{2at}\left(\sigma^n(t) - \phi^n(t)/k_2\right)\phi^n(t)dt. \tag{2.11}$$

Using the positive bound (2.11) in (2.7) and (2.8) gives as a lower bound for the sum of these integrals

$$\left(\sum_{j=1}^{v} c_j e^{ad_j} + \int_0^\infty e^{at} y'(t)dt\right) \int_0^{T_n} e^{2at}(\sigma^n(t)-\phi^n(t)/k_2)\phi^n(t)dt \ . \tag{2.12}$$

Using (2.1) and (2.12) in (2.6) shows that (2.6) holds.

At this point the necessary time domain relationships have been obtained. The next step is to make use of Parseval's Theorem in converting the time domain integrals into corresponding integrals in the frequency domain.

Let $\sigma_\phi(t)$ and $\dot{\sigma}_\phi(t)$ be those components of $\sigma(t)$ and $\dot{\sigma}(t)$, respectively, due to the feedback signal $-\phi(t)$. Then

$$\begin{aligned}
& d \int_0^{T_n} e^{2at} \dot{\sigma}^n(t)\phi^n(t)dt + 2da \int_0^{T_n} e^{2at} \sigma^n(t)\phi^n(t)dt \\
& + c \int_0^{T_n} e^{at}((\delta(t) + y(t)) * [(\sigma^n(t) - \phi^n(t)/k_2)e^{at}]) \phi^n(t)dt = \\
& d \int_0^{T_n} e^{2at} \dot{\sigma}_\phi{}^n(t)\phi^n(t)dt + 2da \int_0^{T_n} e^{2at} \sigma_\phi{}^n(t)\phi^n(t)dt \\
& + c \int_0^{T_n} e^{at} ((\delta(t) + y(t)) * [(\sigma_\phi{}^n(t) - \phi^n(t)/k_2)e^{at}] \phi^n(t)dt \\
& + d \int_0^{T_n} e^{2at} \dot{\sigma}_i{}^n(t)\phi^n(t)dt + 2da \int_0^{T_n} e^{2at} \sigma_i{}^n(t)\phi^n(t)dt \\
& + c \int_0^{T_n} e^{at} ((\delta(t) + y(t)) * [\sigma_i{}^n(t) \ e^{at}]) \phi^n(t)dt \ .
\end{aligned} \tag{2.13}$$

In the first three integrals on the right hand side of (2.13) let $\sigma_\phi^{\,n}(t)$ be replaced by $\sigma_\phi^{\,n*}(t)$ and $\dot{\sigma}_\phi^{\,n}(t)$ by $\dot{\sigma}_\phi^{\,n*}(t)$ where

$$\sigma_\phi^{\,n*}(t) = F^{-1}[-G(j\omega)\ F(\sigma_\phi^{\,n}(t))]$$

and

$$\dot{\sigma}_\phi^{\,n*}(t) = F^{-1}\ [-j\omega G(j\omega)\ F(\sigma_\phi^{\,n}(t))]\ .$$

In the first two integrals since the starred and unstarred quantities are equal in $(0, T_n)$ and since $\phi^n(t)$ is zero outside $(0, T_n)$, this change can be made without altering the values of these integrals. For the third integral the identical reasoning plus $\delta(t) + y(t)$ being zero for $t < 0$ shows that the substitution can be made in this case also without changing the value of the integral. A second desired modification is to replace the $0, T_n$ limits on all 6 of the integrals on the right hand side of (2.13) by $-\infty, \infty$; once again this is justified by the nature of $\phi^n(t)$. This reasoning also allows the last substitution which is to be made in the third integral, namely the replacement of $((\delta(t) + y(t)) * [\sigma_i^{\,n}(t)e^{at}])$ by $((\sigma(t) + y(t)) * [\sigma_i^{\,n}(t)\ e^{at}])^n$. The second function is equal to the first in $(0, T_n)$ and zero elsewhere. With these changes (2.13) becomes

$$d \int_{-\infty}^{+\infty} e^{2at}\ \dot{\sigma}_{\phi}^{n*}(t)\phi^{n}(t)dt + 2da \int_{-\infty}^{+\infty} e^{2at}\ \sigma_{\phi}^{n*}(t)\phi^{n}(t)dt$$

$$+ c \int_{-\infty}^{+\infty} e^{at}((\delta(t) + y(t)) * [(\sigma_{\phi}^{n*}(t) - \phi^{n}(t)/k_2)e^{at}])\phi^{n}(t)dt$$

$$+ d \int_{-\infty}^{+\infty} e^{2at}\ \dot{\sigma}_{i}^{n}(t)\phi^{n}(t)dt + 2da \int_{-\infty}^{+\infty} e^{2at}\ \sigma_{i}^{n}(t)\phi^{n}(t)dt$$

$$+ c \int_{-\infty}^{+\infty} e^{at}\ ((\delta(t) + y(t)) * [\sigma_{i}^{n}(t)e^{at}])^{n}\ \phi^{n}(t)dt\ . \qquad (2.14)$$

Applying the Parseval Theorem to (2.14) and using the fact that only the real parts of the first three integrands give a non-zero contribution to the values of these integrals gives

$$-\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{Re}\ [d(j\omega - a)\ G(j\omega - a) + 2da\ G(j\omega - a)$$

$$+ c\ [1 + Y(j\omega)][G(j\omega - a) + 1/k_2)]\ |F(\phi^{n}(t)e^{at})|^{2}\ d\omega$$

$$+\frac{1}{2\pi} \int_{-\infty}^{+\infty} F[de^{at}\ \dot{\sigma}_{i}^{n}(t) + 2da\ e^{at}\ \sigma_{i}^{n}(t) +$$

$$+ c((\delta(t) + y(t)) * [\sigma_{i}^{n}(t)e^{at}])^{n}]\ \bar{F}(\phi^{n}(t)e^{at})\ d\omega\ . \qquad (2.15)$$

Using c, the first integral can be rewritten as

$$-\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{Re}\ H(j\omega)\ |F(\phi^{n}(t)e^{at})|^{2}\ d\omega \qquad (2.16)$$

and the second as

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} F(p(t))\ \bar{F}(\phi^{n}(t)\ e^{at})\ d\omega \qquad (2.17)$$

where the p(t) is defined in the statement of the theorem. Using the approach given in Aizerman and Gantmacher [13] an upper bound that can be obtained for (2.15) with (2.16) and (2.17) substituted into **it** is

$$\frac{1}{8\pi} \int_{-\infty}^{+\infty} \frac{|F(p(t))|^2}{\mathrm{Re}H(j\omega)} \, d\omega \, . \tag{2.18}$$

From the definition of $P(j\omega)$, $Q(j\omega)$, and m(t) given in the statement of the theorem, an application of Parseval's theorem gives for (2.18)

$$\frac{1}{4} \int_{o}^{\infty} m^2(t)dt \, . \tag{2.19}$$

Using (2.5) on the left hand side of (2.13) together with the bound on the right hand side of (2.13) given by (2.19) results in

$$de^{2aT_n} \Phi(T_n) + 2ad \int_o^{T_n} e^{2at} \sigma(t)\phi(t)dt - 2ad \int_o^{T_n} e^{2at} \Phi(t)dt$$
$$+ c \int_o^{T_n} e^{at}((\delta(t) + y(t)) * [(\sigma^n(t) - \phi^n(t)/k_2)e^{at}])\phi^n(t)dt \leq$$
$$(1/4) \int_o^{\infty} m^2(t)dt + d\Phi(0). \tag{2.20}$$

Since the sum of the second and third integrals is non-negative, the desired bound

$$\phi(T_n) \leq e^{-2aT_n} [\frac{\int_o^{\infty} m^2(t)dt}{4d} + \phi(0)] \tag{2.21}$$

follows . Q.E.D.

Theorem 2.2. Let all of the conditions of theorem 2.1 hold and in addition let $\phi(\sigma)$ be an odd function. Then the assertion of theorem 2.1 holds with y(t) permitted to take on positive as well as negative values.

Proof. The only difference in the proof as compared with that of theorem 2.1 is that in place of (2.6) it is desired to show

$$\int_0^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\phi^n(t)\,dt -$$

$$\left|\int_0^{T_n} e^{at}(y(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at}))\phi^n(t)dt\right| \geq 0 \tag{2.22}$$

To show this, lemma 2 for the odd function case is applied to give

$$\left|\int_0^{T_n} e^{2a(t-\lambda)}(\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\,\phi^n(t)dt\right| \leq$$

$$\int_0^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\,\phi^n(t)dt. \tag{2.23}$$

Using (2.23) in (2.7) and (2.8) gives

$$\left|\int_0^{T_n} e^{at}\,(y(t) * ((\sigma^n(t) - \phi^n(t)/k_2)\,e^{at}))\,\phi^n(t)dt\right| \leq$$

$$[\sum_{j=1}^{v} |c_j|\,e^{ad_j} + \int_0^{\infty} e^{at}|y'(t)|dt]\int_0^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\phi^n(t)dt\;. \tag{2.24}$$

(2.24) shows that (2.22) holds. The remainder of the proof of the theorem is unchanged. Q.E.D.

A Simpler Bound From the Computational Standpoint

It is possible to modify (2.2) in order to obtain a simpler form for computational purposes. As the bound stands, p(t) is zero for $t > T_n$. This means that $\int_0^\infty m^2(t)dt$ has to be calculated for each value of T_n. Rather than using the transform of this truncated p(t) in the development, it is possible to use the Fourier transform of the untruncated function directly independent of T_n. While the original approach should give an improved result for small values of T_n, the latter approach definitely requires less computational effort which is important in hand calculation.

Theorem 2.3. Let the conditions of either theorem 2.1 or theorem 2.2 hold. Then the assertions of these theorems hold with p(t) replaced by

$$p(t) = e^{at}[(c + 2ad)\,\dot{\sigma}_1(t) + d\,\sigma_1(t)] + c(\sigma_1(t)e^{at} * y(t)). \tag{2.25}$$

Proof. Referring to (2.14) it is seen that the change in the definition of p(t) does not affect the value of the last three integrals on the right hand side of this equation. Also, since the new p(t) is Fourier transformable due to G(s) having poles to the left of $s = -a$ and due to (2.1) holding, it follows that the remaining steps in the proof can be carried out without any alteration. Q.E.D.

Example 1. Let $G(s) = \frac{1}{(s + 1)(s + 5)}$, $k_2 = 50$, and $\phi(\sigma)$ be a monotone nonlinearity. It is assumed that this system is excited by a unit impulse input. The Popov criterion shows that this system is asymptotically stable in the large. Since the Popov criterion is applicable, it is reasonable to attempt to satisfy the real part criterion with $Y(j\omega) = 0$. Since the pole of $G(s)$ closest to the origin is -1, a must be chosen less than 1. Let a be chosen arbitrarily as .5. The real part criterion c is then with $c = 1$

$$\mathrm{Re}\left[\frac{(1 + .5d + dj\omega)}{(j\omega+.5)(j\omega+4.5)}\right] + .02 \geq 0 .$$

If d is chosen such that the zero of the term in brackets is located between the two poles, the real part of the first term will be non-negative and c is satisfied. Setting $d = 1$ gives

$$H(s) = \frac{(s + 1.5)}{(s + .5)(s + 4.5)} + .02$$

$Q(j\omega)$ obtained by factoring the reciprocal of the real part of $H(j\omega)$ is

$$\frac{7.07(s + .5)(s + 4.5)}{s^2 + 14.9s + 13.1} .$$

For a unit impulse input $\sigma_1(t) = .25e^{-t} - .25e^{-5t}$ and $\dot{\sigma}_1(t) = -.25e^{-t} + 1.25e^{-5t}$. Then $p(t) = e^{.5t}(2\sigma_1(t) + \dot{\sigma}_1(t)) = .25e^{-.5t} + .75e^{-4.5t}$. $\int_0^\infty m^2(t)dt$ evaluated using Parseval's theorem and tables is 1.86. Substituting this value into the bound expression gives $\Phi(t) \leq .465e^{-t}$.

In order to determine the closeness of this bound for a particular case, let $\phi(\sigma) = 50\sigma$. This choice gives $\Phi(\sigma) = 25\sigma^2$. Using the previously established bound results in $|\sigma(t)| \leq .1365e^{-.5t}$. The actual response of the system with a unit impulse input is $.1475e^{-3t} \sin 6.78t$ which has a maximum magnitude of .081 at t = .17 seconds.

C. <u>Some Considerations in Using the Theorems</u>

At first glance it might appear that the best bound would be obtained by using the largest allowed value of a. However, as the parameter a is increased, the value of the quantity multiplying the exponential term in the bound expression will generally increase since the minimum value of the real part of $H(j\omega)$ will get smaller. With bounds available for different a's, it is of course possible to combine them to get an improved overall bound by taking the smallest bound at a given time.

With regard to the allowed values for a, it has already been stated in the theorem that a must be less than the magnitude of the real part of the pole of $G(j\omega)$ closest to the j axis. By considering the linear case, it is also seen that a must lie to the right of that portion of the root locus of the system corresponding to the gain in the sector $(0, k_2)$.

Once a has been chosen, it is necessary to check the real part criterion to determine whether it is satisfied. Presumably, the asymptotic stability of the system will have been demonstrated so that a candidate for a $Y(s)$ function is available as well as a value of d/c. It is to be noted that the satisfaction of the real part condition only depends upon d/c but that the value of the bound obtained depends upon both these parameters. If the real part condition is not satisfied for this choice for all ω, the parameters can be altered and a new value of $Y(j\omega)$ selected. The required changes in the parameters and $Y(j\omega)$ should be evident from the first try.

It must always be made certain that $\int_0^\infty e^{at} \, |y(t)| dt < 1$.

A point to note is that the larger the value of a, the more difficult it is to satisfy the criterion since ad $G(j\omega - a)$ has a larger coefficient and since the area associated with $y(t)$ becomes less, implying that the maximum phase angle that can be obtained from $1 + Y(j\omega)$ is less than 90°.

Using a computer it is possible to obtain an optimum value for the parameters c and d and for $Y(j\omega)$ by selecting these quantities to minimize the function of time or number multiplying the exponential term in the bound expression. With hand calculation techniques one would have to be satisfied with a few different trials for these quantities.

D. Case of $x(t) \neq 0$

If in order to show stability a multiplier is required which has $z(t) = F^{-1}(Z(j\omega))$ non-zero for $t \leq 0$, the bounding inequality becomes more complicated in that the value of $|\phi^n(t)|$ max, the maximum value of $|\phi^n(t)|$ in the interval $(0, T_n)$, must be used. This result is presented in the next theorem.

Theorem 2.4. For the system of figure 1 excited by initial conditions let a and b of theorem 2.1 hold and let

$$c \ \text{Re}\, H(j\omega) = \text{Re}[c(1 + X(j\omega) + Y(j\omega))(G(j\omega - a) + 1/k_2)$$
$$+ dj\omega\, G(j\omega - a) + ad\, G(j\omega - a)] \geq b > 0 \qquad (2.26)$$

where b, c, and d are positive numbers, $x(t)$ and $y(t)$ are composed of delayed impulses and a piecewise continuous function that satisfy $x(t) = 0$ for $t > 0$, $y(t) = 0$ for $t < 0$, $x(t) \leq 0$ for $t < 0$, $y(t) \leq 0$ for $t > 0$. The magnitude of the piecewise continuous component of $x(t)$ is assumed to be less than $\ell \exp(ft)$ where ℓ and f are positive numbers and

$$\int_{-\infty}^{+\infty} e^{-a|t|}\, |x(t) + y(t)|dt < 1. \qquad (2.27)$$

Then

$$\Phi(T_n) \leq e^{-2aT_n} \left[\frac{\int_o^\infty m^2(t)dt}{4d} + \Phi(0) + M(T_n)|\phi^n(t)|^2_{max}\right]$$

where $m(t) = F^{-1}[P(j\omega)\, Q(j\omega)]$ with

$$p(t) = e^{at}\,[(c + 2ad)\sigma_1{}^n(t) + d\,\dot{\sigma}_1{}^n(t)] +$$

$$c\,[\sigma_1{}^n(t)\,e^{at} * (x(t) + y(t))]^n$$

and $Q(j\omega)$ is defined by $1/\text{Re}\,H(j\omega) = Q(j\omega)\,Q(-j\omega)$.

$$M(T_n) = c \int_0^{T_n} e^{at} \int_{-\infty}^{t} e^{a(t-\lambda)} \int_{t-\lambda-T_n}^{t-\lambda} |g(\varepsilon)|\,|x(\lambda)|\,d\varepsilon\,d\lambda\,dt.$$

where $g(t) = F^{-1}\,(G(j\omega))$.

Proof: The proof is identical with the proof of theorem 2.1 until (2.6) isreached. In place of (2.6) it is to be shown that

$$\int_0^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\,\phi^n(t)dt +$$

$$\int_0^{T_n} e^{at}[x(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at})]\,\phi^n(t)dt +$$

$$\int_0^{T_n} e^{at}\,[y(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at})]\,\phi^n(t)dt > 0\,. \tag{2.28}$$

Let $x(t) = \sum_{i=1}^{u} a_i\,\delta(t + b_i) + x'(t)$ where $x'(t)$ is the piecewise continuous component of $x(t)$. Substituting the impulsive component of $x(t)$ in the second integral above gives

$$\sum_{i=1}^{u} a_i\,e^{ab_i} \int_0^{T_n} e^{2at}\,[\sigma^n(t + b_i) - \phi^n(t + b_i)/k_2]\,\phi^n(t)dt \tag{2.29}$$

and substituting the piecewise continuous component $x'(t)$ into this same integral gives with a change in the order of integration

$$\int_{-\infty}^{0} x'(\lambda)\, e^{-a\lambda} \int_{0}^{T_n} e^{2at}[\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2]\phi^n(t)dt\, d\lambda. \tag{2.30}$$

Writing out $\int_0^{T_n} e^{2at}\,[\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2]\,\phi^n(t)dt$ as in (2.10) gives that this integral is less than or equal to

$$\int_0^{T_n} e^{2at}\,(\sigma_+^n(t-\lambda) - \phi_+^n(t-\lambda)/k_2)\;\phi_+^n(t)dt +$$

$$\int_0^{T_n} e^{2at}(\sigma_-^n(t-\lambda) - \phi_-^n(t-\lambda)/k_2)\;\phi_-^n(t)dt\;. \tag{2.31}$$

Applying lemma 2 for $\lambda < 0$ then gives that

$$\int_0^{T_n} e^{2at}\,[\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2]\;\phi^n(t)dt \le$$

$$\int_0^{T_n} e^{2at}\,[\sigma^n(t) - \phi^n(t)/k_2]\;\phi^n(t)dt\;. \tag{2.32}$$

A lower bound on the second and third integrals of (2.28) is then (2.32) times

$$[\sum_{i=1}^{u} a_i\, e^{ab_i} + \sum_{j=1}^{v} c_i\, e^{ad_i} + \int_{-\infty}^{+\infty} (x'(t)\, e^{-at} + y'(t)e^{+at})]dt$$

which shows that (2.28) holds.

Next, let the term

$$c \int_{o}^{T_n} e^{at} (x(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at})) \phi^n(t)dt$$

be added to (2.13) and let the substitution be made as before. A modification is required in the replacement of $\sigma_\phi{}^n(t)$ by $\sigma_\phi{}^{n*}(t)$ for the added integral on the right hand side of (2.13). For this integral it is necessary to take into account the difference between these two functions due to x(t)'s being non-zero for $t < 0$. Let $\sigma_\phi{}^{n*}(t) = \sigma_\phi{}^n(t) + \sigma_\phi{}^d(t)$. Substituting for $\sigma_\phi{}^n(t)$ according to this expression then gives the following two integrals to be added to (2.14)

$$c \int_{-\infty}^{+\infty} e^{at}(x(t) * [(\sigma_\phi{}^n(t) - \phi^n(t)/k_2)e^{at}]) \phi^n(t)dt$$

$$- c \int_{-\infty}^{+\infty} e^{at}(x(t) * (\sigma_\phi{}^d(t)e^{at})) \phi^n(t)dt . \tag{2.33}$$

An added term involving the initial condition expression is

$$c \int_{-\infty}^{+\infty} e^{at} [x(t) * \sigma_i{}^n(t)e^{at}] \phi^n(t)dt . \tag{2.34}$$

As in the proof of the corresponding stability theorem, the magnitude of the integral involving $\sigma_\phi{}^d(t)$ can be bounded in terms of $|\phi^n(t)|_{max}$. Using the definition of $|\phi^n(t)|_{max}$ and taking absolute magnitudes gives

$$c \int_{-\infty}^{+\infty} e^{at}(x(t) * (\sigma_{\phi}{}^{d}(t)\ e^{at}))\ \phi^n(t)dt \leq$$

$$c\ |\phi^n(t)|^2_{max} \int_0^{T_n} e^{at} \int_{-\infty}^{t} e^{a(t-\lambda)} \int_{t-\lambda-T_n}^{t-\lambda} |g(\varepsilon)||x(\lambda)|d\varepsilon d\lambda dt$$

$$= M(T_n)\ |\phi^n(t)|^2_{max}\ . \tag{2.35}$$

Repeating the steps in (2.15) through (2.19) then gives for (2.20)

$$de^{2aT_n}\ \Phi(T_n) + 2ad \int_0^{T_n} e^{2at}\ \sigma(t)\phi(t)dt - 2ad \int_0^{T_n} e^{2at}\ \Phi(t)dt$$

$$+ c \int_0^{T_n} e^{at}\ ((\delta(t) + x(t) + y(t)) * [(\sigma^n(t) - \phi^n(t)/k_2)e^{at}])\ \phi^n(t)dt$$

$$\leq \frac{1}{4} \int_0^{\infty} m^2(t)dt + d\Phi(0) + M(T_n)\ |\phi^n(t)|^2_{max}\ . \tag{2.36}$$

Then (2.21) becomes

$$\Phi(T_n) \leq e^{-2aT_n} \left[\frac{\int_0^{\infty} m^2(t)dt}{4d} + \Phi(0) + \frac{M(T_n)}{d}|\phi^n(t)|^2_{max}\right]. \tag{2.37}$$

Q.E.D.

Theorem 2.4 can be applied in the case where $\phi(\sigma)$ is an odd monotone function with $x(t)$ and $y(t)$ being less restricted. The proof is similar to that of theorem 2.1 so it will not be repeated here.

Theorem 2.5. Let all of the conditions of theorem 2.4 hold and in addition let $\phi(\sigma)$ be an odd function. Then the assertion of theorem 2.4 holds with $x(t)$ and $y(t)$ permitted to take on positive as well as negative values

Although $M(T_n)$ is independent of system excitation as developed in the proof of the theorem, this is not the case for $|\phi^n(t)|_{max}$. A value must be obtained for this quantity before the bound can be applied. The simplest way to find this quantity is by using theorems 2.4 or 2.5 with a = 0. T_n is chosen as that value of time at which $|\phi^n(t)|_{max}$ occurs. Then by using the fact that $\Phi(T_n)$ approaches infinity more rapidly than $|\phi(\sigma)|^2$, a bound can be obtained on $|\phi|$ by finding the value of this variable above which the bounding inequality does not hold.

E. A Different Bound

The bound (2.2) given by theorem 2.1 as well as the other bounds obtained thus far depend upon the square of the initial condition excitation. As long as $\phi(\sigma)$ is in its linear range, a reasonable bound is obtained for σ. To see this, let $\Phi(\sigma) = c_1\sigma^2$ where c_1 is a positive number. In the calculation of the bound for σ a square root must be taken and σ is then effectively bounded by a linear function of the initial conditions. On the other hand if $\phi(\sigma)$ is in a saturation region, $\Phi(\sigma) = c_2|\sigma| + c_3$, resulting in the bound depending upon the square of the initial conditions. To try to get a better estimate in this saturation case, the approach employed by Lefschetz [11] will be used rather than the "completing the square" approach given in Aizerman and Gantmacher [13] that has been utilized thus far. The Lefschetz approach yields a bound dependent upon the magnitude of the initial conditions.

Theorem 2.6 Let all of the conditions of either theorems 2.1 or 2.2 hold. Then another bound on $\Phi(T_n)$ is

$$\Phi(T_n) \leq e^{-2aT_n} \left[\frac{|\phi^n(t)|_{max} \int_0^\infty |p(t)|dt}{d} + \Phi(0) \right] \tag{2.38}$$

where

$$p(t) = e^{2at}[(c + 2ad)\, \sigma_i^{\,n}(t) + d\, \dot{\sigma}_i^{\,n}(t)] + c\, e^{at}[\sigma_i^{\,n}(t)e^{at} * (y(t))]^n \tag{2.39}$$

Proof. The proof is unchanged until (2.15) is reached. At this point, since (2.16) is negative, it can be dropped and the second integral (2.17) retained. Then, the left hand side of (2.20) is less than or equal to the magnitude of (2.17) written in time domain form which is

$$c \int_{-\infty}^{+\infty} e^{at}((\delta(t) + y(t)) * [\sigma_i^{\,n}(t)e^{at}])^n\, \phi^n(t)dt +$$

$$d \int_{-\infty}^{+\infty} e^{2at}\, \dot{\sigma}_i^{\,n}(t)\, \phi^n(t)dt + 2da \int_{-\infty}^{+\infty} e^{2at}\, \sigma_i^{\,n}(t)\, \phi^n(t)dt. \tag{2.40}$$

The magnitude of this integral is less than or equal to

$$|\phi^n(t)|_{max} \int_{-\infty}^{+\infty} |p(t)|dt\ . \tag{2.41}$$

where p(t) is defined above. With the exception of the use of the new bound, the remainder of the proof is unchanged. Q.E.D.

In a similar way theorems 2.4 and 2.5 can be restated using this new bound. The modification in the proof is identical to that given for theorem 2.6.

Theorem 2.7. Let all of the conditions of either theorems 2.4 or 2.5 hold. Then another bound on $\Phi(T_n)$ is

$$\Phi(T_n) \leq e^{-2aT_n} \left[\frac{|\phi^n(t)|_{max} \int_0^\infty |p(t)|dt}{d} + \frac{M(T_n)}{d} |\phi^n(t)|^2_{max} + \Phi(0) \right] \tag{2.42}$$

where

$$p(t) = e^{2at} \left[(c + 2ad)\, \sigma_i^{\,n}(t) + d\, \dot{\sigma}_i^{\,n}(t)\right] + c\, e^{at}[\sigma_i^{\,n}(t) e^{at} * (x(t) + y(t))]^n \tag{2.43}$$

and $M(T_n)$ is defined in the statement of theorem 2.4.

Example 2. Consider the same problem as that of example 1 and let the nonlinear characteristic be a saturation function defined by $\phi(\sigma) = 50\sigma$ for $0 \leq |\sigma| \leq .02K$ and $\phi(\sigma) = \pm K$ for $.02K \leq |\sigma| < \infty$ with the + sign applying for positive values of σ and the - sign for negative values. Using (2.39) and the previously computed values of $\sigma_i(t)$ and $\dot{\sigma}_i(t)$ gives $p(t) = .25 + .75e^{-4t}$. The bound is then $\Phi(T_n) \leq K\,(.25T_n\, e^{-T_n} - .1875e^{-5T_n} + .1875e^{-T_n})$ with $|\phi^n(t)|_{max} = K$. The bounds for σ are then

$$|\sigma| \leq .25\, T_n\, e^{-T_n} - .1875e^{-5T_n} + .1875e^{-T_n} + .01K, |\sigma| \geq .02K$$

$$\sigma^2 \leq .04K(.25T_n\, e^{-T_n} - .1875e^{-5T_n} + .1875e^{-T_n}),\ |\sigma| \leq .02K\ .$$

Plots of this bound (called the L bound) and of the bound computed in example 1 (called the AG bound) are plotted in figures 7-10 for various values of the saturation level K. The smaller the value of K, the better the results of the L bound as compared with the AG bound.

F. A Response Bound With an External Input Applied

The introduction of the e^{at} multiplier for $\Phi(T_n)$ allows a bound to be obtained for the response of the system with certain external inputs applied. Theoretically, it is only necessary to make certain that the input is such that piecewise continuity and Fourier transformability are guaranteed for certain pertinent functions. From the practical standpoint some difficulty may be encountered in finding a bound for $|\phi^n(t)|_{max}$ in theorems 2.4, 2.5, 2.6, and 2.7. If $\int_{-\infty}^{\infty} |p(t)|dt$ is bounded for a = 0, a bound can be computed as discussed previously; if this integral is not bounded, it is necessary to calculate a time varying bound for $|\phi^n(t)|^2_{max}$ using the theorems with a = 0 and choosing $|\phi^n(t)|^2_{max}$ as occurring at t = T_n as the worst case. Since $|\phi^n(t)|_{max}$ does not appear in theorems 2.1 and 2.2, these theorems can be applied with no change in the computation procedure. Examples of possible inputs include a sinusoidal function, a ramp function and an exponential function. This discussion is summarized in the following theorem.

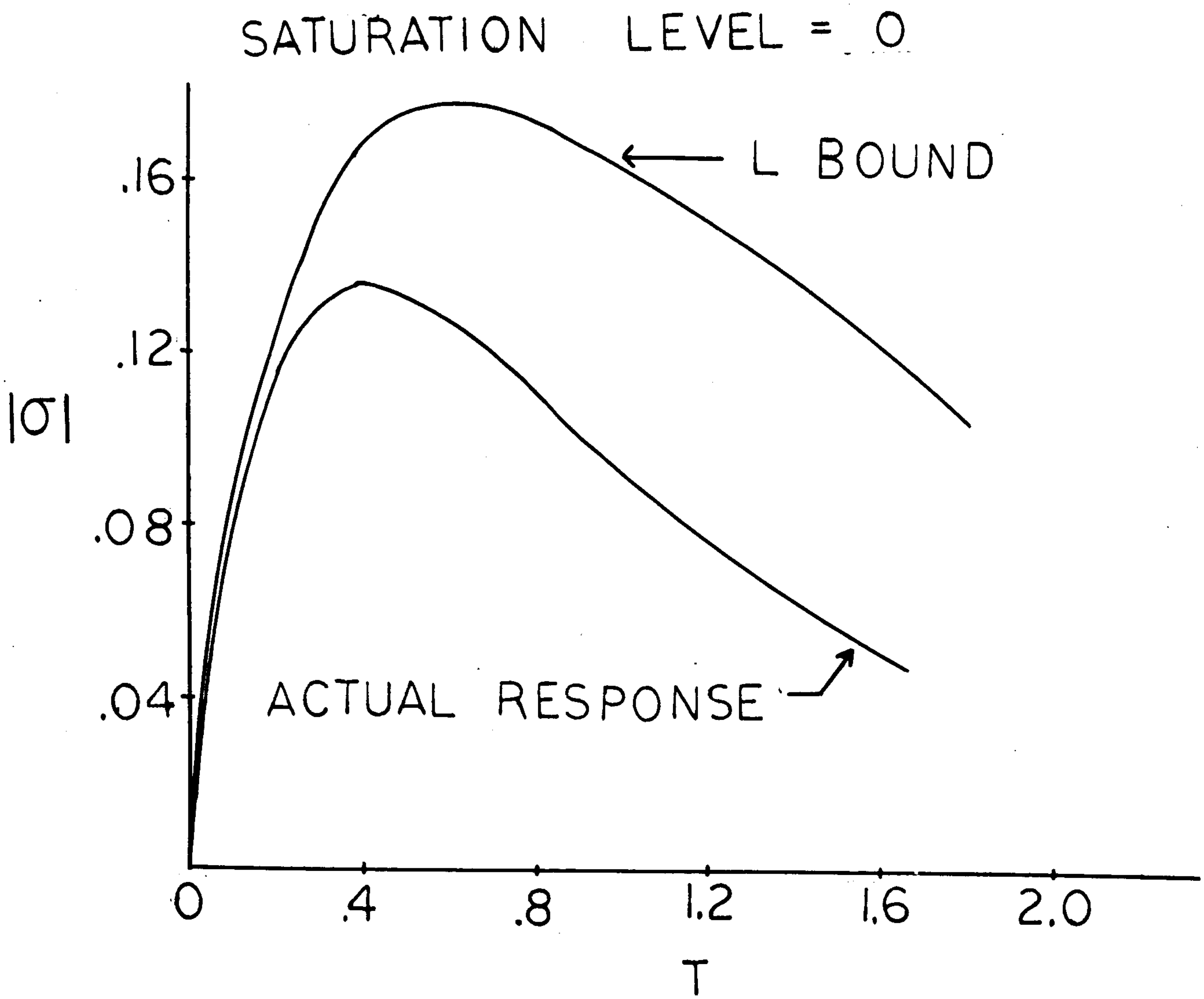

Figure 7. Bound on σ for the saturation level k = 0.

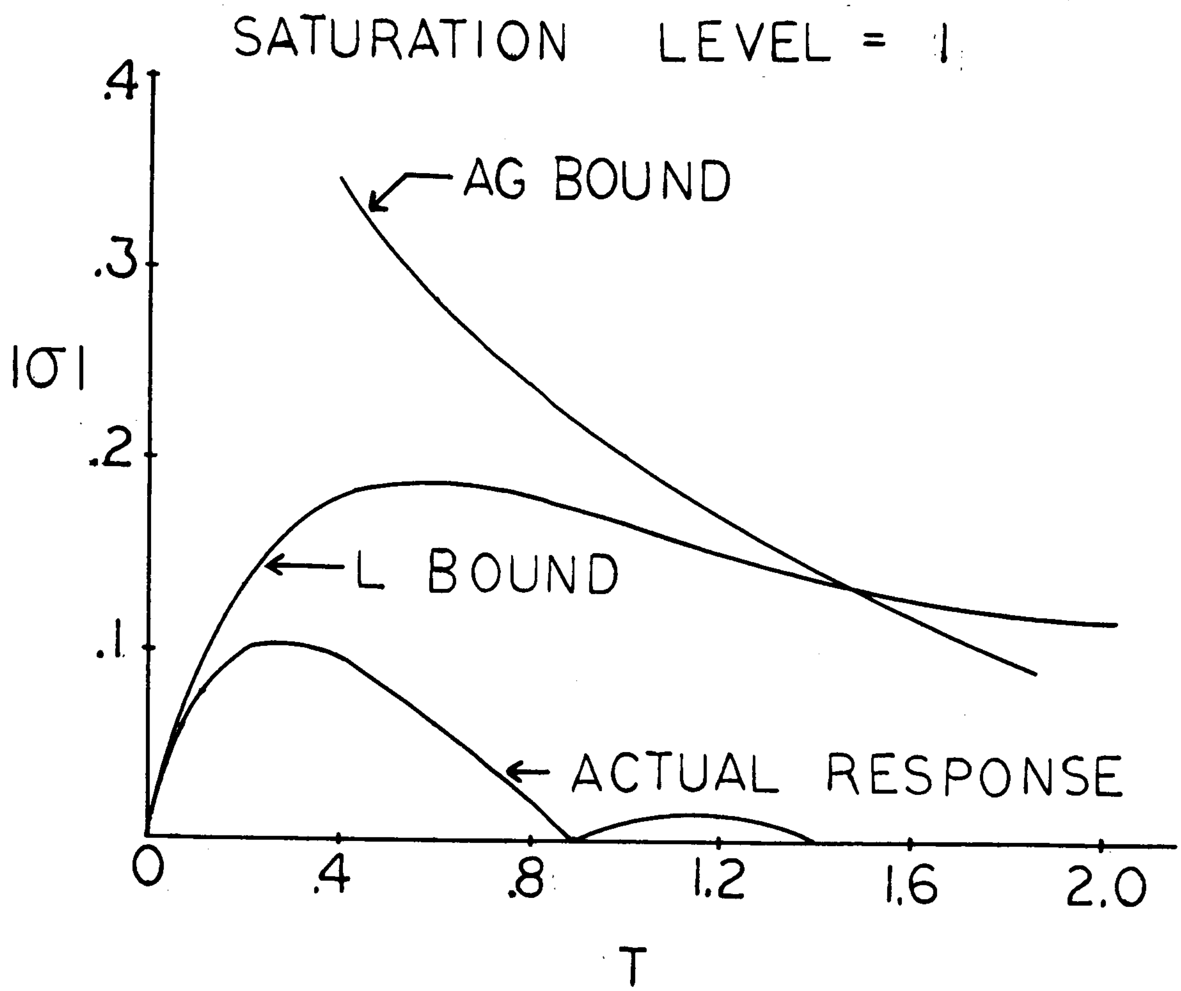

Figure 8. Bounds on σ for the saturation level $k = 1$.

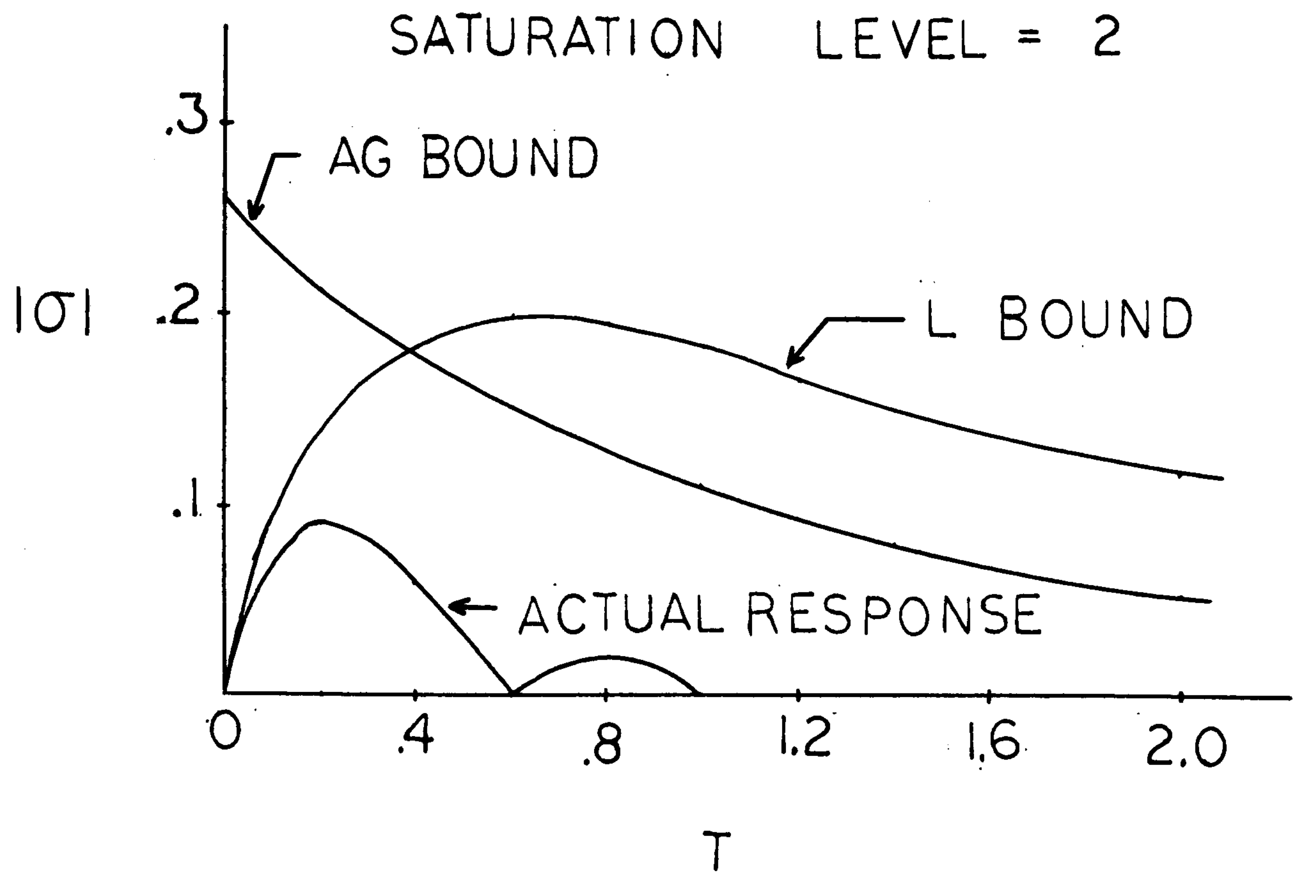

Figure 9. Bounds on σ for the saturation level k = 2.

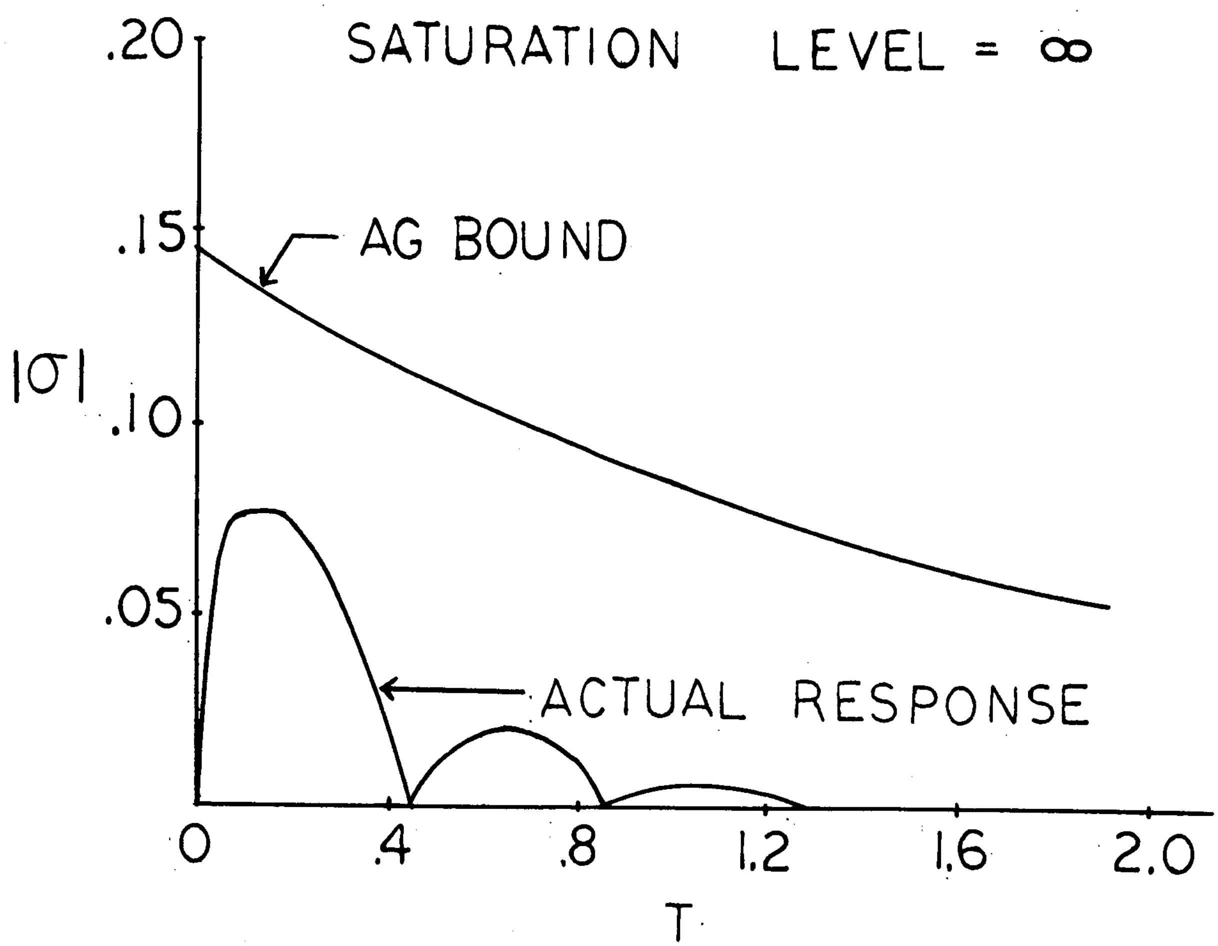

Figure 10. Bounds on σ for the saturation level $k = \infty$ (linear case).

Theorem 2.8. Let the conditions of either theorem 2.1, 2.2, 2.4, 2.5, 2.6, or 2.7 hold. If the input to the system is such that $\sigma^n(t)$, $\dot{\sigma}^n(t)$, $\phi^n(t)$, $\sigma_r{}^n(t)$, and $\dot{\sigma}_r{}^n(t)$ are Fourier transformable, the assertions of these theorems hold with $\sigma_i{}^n(t)$ and $\dot{\sigma}_i{}^n(t)$ replaced by $\sigma_i{}^n(t) + \sigma_r{}^n(t)$ and $\dot{\sigma}_i{}^n(t) + \dot{\sigma}_r{}^n(t)$ respectively. $\sigma_r{}^n(t)$ and $\dot{\sigma}_r{}^n(t)$ are equal to those components of $\sigma(t)$ and $\dot{\sigma}(t)$, respectively, due to the direct action of the input (the input acting through $G(s)$) in $(0, T_n)$ and zero outside this interval.

Example 3. Let $G(s) = 1/(s + 1)$, $k_2 = 10$, the nonlinearity be monotone, and the excitation be an input of sin t with the initial conditions zero. This $G(s)$ is sufficiently simple that theorem 1 can be applied with $y(t) = 0$.

$$\text{Re } H(j\omega) = \frac{\text{Re } (c + ad + dj\omega)}{j\omega - a + 1} + .1 \,.$$

Set $a = .25$, $c = 1$, and $d = 2$. This then gives $\text{Re}H(j\omega) = 2.1$. $\sigma_r(t) = .5e^{-t} + .707 \cos(t - 135^{\circ})$, $\dot{\sigma}_r(t) = -.5e^{-t} - .707 \sin(t - 135^{\circ})$, and $p(t) = 2e^{.25t} \sin t$. Using these quantities then gives as the bound

$$\Phi(T_n) \leq .238 - .014 \cos 2T_n - .0561 \sin 2T_n - .2235e^{-.5T_n} \leq .2959.$$

For the special case $\phi(\sigma) = 10\sigma$, using the above bound gives $\sigma(T_n) \leq .243$. The actual response is $\sigma(t) = .0082e^{-11t} + .0905 \cos(t - 95.2^{\circ})$.

Example 4. Let the system be the same as in example 3 but let the input be a unit ramp rather than a sinusoidal input. $\sigma_r(t) = t-1+e^{-t}$, $\dot{\sigma}_r(t) = 1 - e^{-t}$, $p(t) = 2e^{.25t}t$. Using these values gives

$$\phi(T_n) \leq 1.91(.25\, T_n^{\,2} - T_n + 2) - 3.82e^{-.5T_n}$$

from which it is seen that the bound approached for large T_n is $.4775\, T_n^{\,2}$. With $\phi(\sigma) = 10\sigma$, this gives as a bound for large T_n $|\sigma| \leq .309\, T_n$. The actual response for large values of T_n is $\sigma = .0909\, T_n$.

For both of these examples by referring to [6] - [9] and treating the inputs as being zero outside $(0, T_n)$, it can be shown that the conditions of the theorem are satisfied.

As was pointed out in the introduction, the application of this theorem can show Liapunov stability with certain inputs applied. The case of example 3 with the sinusoidal input applied illustrates this point.

G. Modification For the Case of Poles to The Left of the Line s = -a

In the case of a system in which a lag compensator has been incorporated in order to increase the gain of the system at low frequencies, the significant portions of the response are usually characterized by one time constant while another time constant due to the lag compensator characterizes the response for large values of time. In the theorems discussed thus far, it has been assumed that a is less than the magnitude of the real part of the pole closest to the origin. Therefore, these theorems would only be able to yield a bound that would be realistic for large t. The theorem below allows the calculation of a bound that should give good results for the significant portions of the response of these systems. The approach used is basically one in which the given G(s) is replaced by another transfer function equal to g(t) in $(0, T_n)$ but different from g(t) outside this interval. This modification allows the original theorems to be applied to give a bound valid in the time interval $(0, T_n)$.

Theorem 2.9. Let

$$G(s) = G_1(s) + \sum_{i=1}^{n} \frac{a_i}{s + b_i}$$

$$s\,G(s) = G_2(s) + \sum_{i=1}^{n} \frac{c_i}{s + d_i}$$

where $a > b_i$ but less than the magnitudes of the real parts of the poles of $G_1(s)$ and $G_2(s)$. Then if conditions a and b are satisfied and the modified c given below is also satisfied

$$\text{Re } H(j\omega) = \text{Re}[c(1 + Y(j\omega) + X(j\omega))\,(G_A(j\omega - a) + 1/k_2)$$

$$+ d\,G_B(j\omega - a) + 2ad\,G_A(j\omega - a)] \geq \delta > 0$$

where $X(j\omega)$ may be zero, and

$$G_A(j\omega - a) = G_1(j\omega - a) + \sum_{i=1}^{n} \frac{a_i[1 - e^{(a-b_i)T_n} e^{-j\omega T_n}]}{j\omega - a + b_i}$$

$$G_B(j\omega - a) = G_2(j\omega - a) + \sum_{i=1}^{n} \frac{c_i[1 - e^{(a-b_i)T_n} e^{-j\omega T_n}]}{j\omega - a + b_i}$$

the assertions of theorems 2.1, 2.2, and 2.6 hold without any changes and the assertions of theorems 2.4, 2.5, and 2.7 hold with the $g(\varepsilon)$ used in the definition of $M(T_n)$ replaced by $g_A(\varepsilon) = F^{-1}[G_A(j\omega)]$.

<u>Proof</u>. The only change required in the proof of the theorems is in the step just before the application of Parseval's theorem by which the time domain integrals are converted to frequency domain integrals. $\sigma_\phi^{n*}(t)$ and $\dot{\sigma}_\phi^{n*}(t)$ are redefined as $\sigma_\phi^{n*}(t) = -F^{-1}[G_A(j\omega)\, F(\phi^n(t))]$ and $\dot{\sigma}_\phi^{n*}(t) = -F^{-1}[G_B(j\omega)\, F(\phi^n(t))]$. If $x(t) = 0$, these changes do not alter the values of the integrals in which they appear since these two time functions are equal to $\sigma(t)$ and $\dot{\sigma}(t)$, respectively, in $(0, T_n)$. For $x(t) \neq 0$, the substitutions result in a different value for $\sigma_\phi^d(t)$ but the same steps in the proof are applicable with $g(\varepsilon)$ being replaced by $g_A(\varepsilon)$ in the definition of $M(T_n)$. The reason for the changes is that with the original definitions, $\sigma_\phi^{n*}(t)$ and $\dot{\sigma}_\phi^{n*}(t)$ when multiplied by e^{at} were not Fourier transformable. The new definitions result in Fourier transformable functions when multiplied by the exponential. The other steps in the proof are unchanged.

Example 5. Let $G(s) = -1.005/(s + 2) + 1/(s + 1) + .005/(s + .1)$ for a system with a monotone nonlinearity and a gain $k_2 = 10$. Then $s\,G(s) = 2.010/(s + 2) - 1/(s + 1) - .0005/(s + .1)$. Let $a = .5 = T_n$. This $G_A(s)$ is sufficiently simple that theorem 2.1 can be applied with $Y(s) = 0$, $c = 1$, and $d = 1$. The real part criterion is then

$$\mathrm{Re}\,[2\,G_A(j\omega - a) + G_B(j\omega - a) + .1] =$$

$$1/(j\omega + .5) + .0095(1 - e^{+.4T_n}\,e^{-j\omega T_n})/(j\omega - .4) + .1.$$

The maximum magnitude of the second term on the right hand side is .053. Therefore, $\mathrm{Re}\,H(j\omega) \geq .047$. For convenience in the calculation of the lower bound, this number will be used rather than the actual function of frequency. $p(t) = e^{.5t}[2\sigma_i{}^n(t) + \dot{\sigma}_i{}^n(t)] = e^{-.5t} + .0095e^{+.4t}$ for an impulse input. The bound is then

$$\Phi(T_n) \leq 5.32\,[1.19e^{-T_n} + .000113\,e^{-.2T_n} - e^{-2T_n} - .19\,e^{-1.1T_n}]$$

for $T_n \leq .5$. For $T_n \geq .5$, the bound given by the original theorems can be used with $a < .1$.

H. A Result for the Linear Case

If $\phi(\sigma)$ is a linear function or a nonlinear function in its linear range, it is possible to get an improved result for the frequency domain condition c. To see this, let $\phi(\sigma) = K\sigma$ where $0 < K < \infty$. Then (2.4) can be replaced by

$$a \int_o^{T_n} e^{2at}\, \sigma(t)\, \phi(t)dt - 2a \int_o^{T_n} e^{2at}\, \Phi(t)dt \geq 0$$

since $\sigma(t)\phi(t) = K\sigma^2(t)$ and $\Phi(t) = K\sigma^2(t)/2$. Also, for the linear case

$$\int_o^{T_n} e^{at}\, e^{a(t-\lambda)}\, (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\phi^n(t)dt =$$

$$K(1 - K/k_2) \int_o^{T_n} e^{at}\, \sigma^n(t) e^{a(t-\lambda)} \sigma^n(t-\lambda)dt \leq$$

$$.5K(1 - K/k_2) \left[\int_o^{T_n} e^{2at} \sigma^{n2}(t)dt + \int_o^{T_n} e^{2a(t-\lambda)} \sigma^{n2}(t - \lambda)dt\right] \leq$$

$$K(1 - K/k_2) \int_o^{T_n} e^{2at} \sigma^{n2}(t)dt = \int_o^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\, \phi^n(t)dt$$

which means that the integral magnitude condition can be relaxed. Combining these two results gives for the frequency domain condition c

$$\text{Re } H(j\omega) =$$

$$\text{Re}[(1 + dj\omega + Y(j\omega) + X(j\omega))(G(j\omega - a) + 1/k_2)] \geq \delta > 0$$

where $\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)dt < 1.$

Because of this improved condition, it is possible to choose larger values of the parameter a for the linear case than for the nonlinear case. This suggests the following approach. When $|\sigma(t)|$ is such that the system is in its nonlinear region, one of the bounds already discussed can be claculated. When according to this bound the system is in and remains in the linear region for all succeeding values of t, say $t \geq T^1$, an improved bound is calculated using the real part criterion given above. In applying the theorem this second time, a value is immediately available for $\Phi(T^1)$. However, since $\sigma_i(t)$ and $\dot{\sigma}_i(t)$ are not known for this second application of the theorem, bounds for these two quantities must be calculated using the bound on $\phi(\sigma)$ determined in the first application of the theorem.

I. Conclusion

This chapter has presented a number of different results for bounds on the response of the single nonlinearity time invariant system. The usefulness of these bounds appears to be in two applications. First, it is possible to develop an approach for carrying out an analytical design for a nonlinear system. If the system is excited by initial conditions or by an impulse or step input which can be converted to equivalent initial condition inputs, the theorems given can be used to calculate a bound on $|\sigma(t)|$. Since the desired equilibrium state for the excitation under discussion is the origin, it is possible to obtain a satisfactory design for the response time of the system by adjusting the parameters of the system or by adding a compensator such that the bound on the system output meets the system specifications. Secondly, if a bounded time varying input is applied to the system, it is possible to show Liapunov stability by applying the bounding theorems.

Therefore, the bounding theorems give sufficient conditions for Liapunov stability with a bounded input applied, provided that no common factors of G(s) in the right half s plane or on the $j\omega$ axis have been cancelled.

J. Appendix

Lemma 1. Let $f_a(t)$ and $f_b(t)$ be two continuous functions of t that are zero outside the interval $(0, n\Delta t)$ where n is a positive integer and Δt is a positive number, $f_a(t)\, f_b(t) \geq 0$, $f_a(t) = h(f_b(t))$ where h is a piecewise continuous monotone increasing function of its argument, then if either both $f_a(t)$ and $f_b(t)$ are always non-positive, or non-negative or if h is an odd function with $h(0) = 0$,

$$\sum_{k=o}^{n} |f_a(k\Delta t)\; f_b(k\Delta t - \lambda)| \leq \sum_{k=o}^{n} f_a(k\Delta t)\; f_b(k\Delta t)$$

where λ is a real number such that $|\lambda|/\Delta t$ is an integer.

Proof. The proof of this lemma follows from the proof of the lemma given at the end of chapter 1 in which this result is obtained as an intermediate step.

Lemma 2. Let $f_a(t)$ and $f_b(t)$ be two continuous functions of time that are zero outside the interval $(0, T_n)$ where T_n is a positive number, $f_a(t)\ f_b(t) \geq 0$, $f_a(t) = h(f_b(t))$ where h is a piecewise continuous monotone increasing function of its argument, then if either both $f_a(t)$ and $f_b(t)$ are always non-positive or non-negative or if h is an odd function with $h(0) = 0$,

$$\left| \int_o^{T_n} e^{2a(t-\lambda)}\ f_a(t) f_b(t-\lambda) dt \right| \leq \int_o^{T_n} e^{2at}\ f_a(t) f_b(t) dt, \quad \lambda > 0$$

and

$$\left| \int_o^{T_n} e^{2at}\ f_a(t) f_b(t-\lambda) dt \right| \leq \int_o^{T_n} e^{2at}\ f_a(t) f_b(t) dt, \quad \lambda < 0.$$

Proof. Let Δt be chosen such that $|\lambda|/\Delta t$ is a positive integer and n is the largest integer less than or equal to $T_n/\Delta t$. It is assumed that $|\lambda| < T_n$ for if $|\lambda| \geq T_n$, the assertion of the lemma follows at once. With $\lambda > 0$, let the two summations

$$\sum_{k=o}^{n} \left| f_a(k\Delta t)\ f_b(k\Delta t - \lambda) \right|\ e^{2a(k\Delta t - \lambda)}\ \Delta t \tag{A1}$$

and

$$\sum_{k=o}^{n} f_a(k\Delta t)\ f_b(k\Delta t) e^{2a(k\Delta t)}\ \Delta t \tag{A2}$$

be formed. (A1) divided by Δt may be rewritten as

$$[|f_a(\lambda)\ f_b(0)| + |f_a(\lambda+\Delta t)\ f_b(\Delta t)| + |f_a(\lambda+ 2\Delta t)\ f_b(2\Delta t)| + \dots$$

$$+ |f_a(T_n-\Delta t)\ f_b(T_n- \lambda-\Delta t)| + |f_a(T_n)\ f_b(T_n-\lambda)|]$$

$$+ (e^{2a\Delta t} - 1)\ [|f_a(\lambda+\Delta t)\ f_b(\Delta t)| + |f_a(\lambda+ 2\Delta t)\ f_b(2\Delta t)| + \dots$$

$$|f_a((n-1)\Delta t)\ f_b((n-1)\Delta t - \lambda)| + |f_a(n\Delta t)\ f_b(n\Delta t-\lambda)|]$$

$$+ (e^{4a\Delta t} - e^{2a\Delta t})[|f_a(\lambda+ 2\Delta t)\ f_b(2\Delta t)| + |f_a(\lambda+3\Delta t)\ f_b(3\Delta t)| + \dots$$

$$|f_a((n-1)\Delta t)\ f_b((n-1)\Delta t-\lambda)| + |f_a(n\Delta t)\ f_b(n\Delta t-\lambda)|]$$

$$+ \dots$$

$$+ (e^{2a(\lambda-\Delta t)} - e^{2a(\lambda-2\Delta t)})\ [|f_a((n-1)\Delta t)\ f_b((n-1)\Delta t-\lambda)| +$$

$$|f_a(n\Delta t)\ f_b(n\Delta t-\lambda)|]$$

$$+ (e^{2a\lambda} - e^{2a(\lambda-\Delta t)})\ |f_a(n\Delta t)\ f_b(n\Delta t-\lambda)|. \qquad \text{(A3)}$$

Similarly, (A2) divided by Δt may be rewritten as

$$[f_a(0)\ f_b(0) + f_a(\Delta t)\ f_b(\Delta t) + f_a(2\Delta t)\ f_b(2\Delta t) + \dots$$

$$+ f_a((n-1)\Delta t)\ f_b((n-1)\Delta t) + f_a(n\Delta t)\ f_b(n\Delta t)]$$

$$+ (e^{2a\Delta t} - 1)[f_a(\Delta t)\ f_b(\Delta t) + f_a(2\Delta t)\ f_b(2\Delta t) + \dots + f_a(n\Delta t)\ f_b(n\Delta t)]$$

$$+ (e^{4a\Delta t} - e^{2a\Delta t}) \, [f_a(2\Delta t) \, f_b(2\Delta t) + f_a(3\Delta t) \, f_b(3\Delta t) + \ldots \, f_a(n\Delta t) \, f_b(n\Delta t)]$$

$$+ \ldots$$

$$+ (e^{2a\lambda} - e^{2a(\lambda-\Delta t)}) [f_a(\lambda) f_b(\lambda) + f_a(\lambda+\Delta t) f_b(\lambda+\Delta t) + \ldots \, f_a(n\Delta t) \, f_b(n\Delta t)]$$

$$+ \ldots$$

$$+ (e^{2an\Delta t} - e^{2a(n-1)\Delta t}) \, [f_a(n\Delta t) \, f_b(n\Delta t)] \, . \qquad \text{(A4)}$$

Comparing the terms in (A3) and (A4) having the same exponential multiplier and using lemma 1 on the terms of (A3), it follows that (A3) is less than or equal to (A4). Since

$$\left| \int_o^{T_n} f_a(t) \, f_b(t-\lambda) \, e^{2a(t-\lambda)} \, dt - \sum_{k=o}^{n} \, f_a(k\Delta t) \, f_h(k\Delta t-\lambda)\Delta t \right|$$

$$< \varepsilon(\Delta t)$$

where $\varepsilon(\Delta t)$ is a positive number whose value depends upon Δt, taking the limit as $\Delta t \to 0$ gives

$$\left| \int_o^{T_n} f_a(t) \, f_b(t-\lambda) \, e^{2a(t-\lambda)} dt \right| \le \int_o^{T_n} \, f_a(t) \, f_b(t) \, e^{2at} \, dt.$$

which is one half of the lemma.

With $\lambda < 0$ the summation

$$\sum_{k=o}^{n} |f_a(k\Delta t) \, f_b(k\Delta t-\lambda)| e^{2ak\Delta t} \, \Delta t \qquad \text{(A5)}$$

is formed and rewritten as

$$[|f_a(0)\ f_b(-\lambda)| + |f_a(\Delta t)\ f_b(\Delta t-\lambda)| + |f_a(2\Delta t)\ f_b(2\Delta t-\lambda)| + \dots$$

$$|f_a(n\Delta t+\lambda)\ f_b(n\Delta t)|]$$

$$+ (e^{2a\Delta t} - 1)[|f_a(\Delta t)\ f_b(\Delta t-\lambda)| + |f_a(2\Delta t)\ f_b(2\Delta t-\lambda)| + \dots$$

$$|f_a(n\Delta t+\lambda)\ f_b(n\Delta t)|]$$

$$+ (e^{4a\Delta t} - e^{2a\Delta t})\ [|f_a(2\Delta t)\ f_b(2\Delta t-\lambda)| + |f_a(3\Delta t)\ f_b(3\Delta t-\lambda)| + \dots$$

$$|f_a(n\Delta t+\lambda)\ f_b(n\Delta t)|]$$

$$+ \dots$$

$$+ (e^{2a(n\Delta t+\lambda)} - e^{2a((n-1)\Delta t+\lambda)})\ [|f_a(n\Delta t+\lambda) f_b(n\Delta t)|]\ . \qquad \text{(A6)}$$

Repeating the foregoing reasoning with (A6) replacing (A3) gives

$$\left| \int_o^{T_n} e^{2at}\ f_a(t)\ f_b(t-\lambda)dt \right| \leq \int_o^{T_n} e^{2at}\ f_a(t)\ f_b(t)dt\ .$$

Q.E.D.

V CHAPTER III. SYSTEM WITH A TIME VARYING NONLINEARITY, SAMPLED DATA SYSTEMS AND SYSTEMS WITH MULTIPLE NONLINEARITIES

A. Time Varying Nonlinearity

The theorem given below is a modification of theorem 1.2 with the modification added to take into account the $\int_0^{T_n} \dot{\sigma}(t)\phi(t)dt$ term no longer being an exact integral. There are a number of ways in which this could be done; the approach used has the merit that it is not necessary to take into account the rate at which the nonlinearity changes with time. Therefore, this theorem appears to be the most generally applicable one that could be developed.

Pertinent references include the works by Sandberg [18] and Rekasius and Rowland [19]. The criteria which are developed in these references do not include anything as general as the Z(s) multiplier used in theorem 3.1.

Theorem 3.1. For the system of figure 1 with ϕ being a time varying nonlinearity let the following conditions hold:

a. $A\ \phi_m(\sigma)\sigma \leq \phi(\sigma,t)\sigma \leq B\ \phi_m(\sigma)\sigma$ where A and B are real numbers satisfying $0 < A \leq 1$ and $1 \leq B < \infty$, $\phi(0,t) = \phi_m(0) = 0$, $\sigma\ \phi(\sigma,t) < k\ \sigma^2$ where $k > 0$ and $\sigma\ \phi_m(\sigma) > 0$ for $\sigma \neq 0$, $d\phi(\sigma,t)/d\sigma$ is a continuous function of σ, $\phi_m(\sigma)$ is a continuous monotone increasing function of σ having an odd part $\phi_{mo}(\sigma)$ that satisfies $|\phi_m(\sigma)| \leq C|\phi_{mo}(\sigma)|$ and $|\phi_{mo}(\sigma)| \leq D|\phi_m(\sigma)|$.

b. Conditions b and c of theorem 1.1.

Then a sufficient condition for asymptotic stability in the large is that

$$\mathrm{Re}[Z(j\omega)\ G(j\omega) + E(G(j\omega) + 1/k) - \alpha(\tfrac{B-A}{2A})\ (k^2 + \omega^2)\ |G(j\omega)|^2] \geq \delta > 0 \quad (3.1)$$

for all real ω where E is a non-negative number, δ is a positive number, and

$$Z(j\omega) = 1 + \alpha j\omega + X(j\omega) + Y(j\omega) \tag{3.2}$$

and

$$\frac{BCD}{A} \left[\int_{-\infty}^{+\infty} (x'^{+}(t) + y'^{+}(t))dt + \sum a_i^{+} + \sum c_i^{+} \right] -$$

$$\frac{B}{A} \left[\int_{-\infty}^{+\infty} (x'^{-}(t) + y'^{-}(t))dt + \sum a_i^{-} + \sum c_i^{-} \right] < 1 \tag{3.3}$$

where $x'^{+}(t)$, $y'^{+}(t)$, a_i^{+}, and c_i^{+} are the positive portions or values of the corresponding non-superscripted functions or numbers and $x'^{-}(t)$, $y'^{-}(t)$, a_i^{-}, and c_i^{-} are the negative portions or values of the corresponding non-superscripted functions or numbers. The magnitude of the piecewise continuous component of $x(t)$ is assumed to be less than $\ell \exp(ft)$ where ℓ and f are positive numbers.

Proof. The proof is identical with the proof of theorem 1.2 except for the handling of the $\int_0^{T_n} \dot{\sigma}(t)\, \phi(t)dt$ term. Because of condition a of the theorem, it is possible to express $\phi(\sigma,t)$ as

$$\phi(\sigma,t) = A\phi_m(\sigma) + \phi_2(\sigma,t) \tag{3.4}$$

where

$$|\phi_2(\sigma,t)| \leq (B-A)\ |\phi_m(\sigma)| \leq \frac{B-A}{A}\ |\phi(\sigma,t)| \tag{3.5}$$

Using this result, it is desired to show that

$$\int_o^{T_n} \dot{\sigma}(t)\phi_2(\sigma,t)dt + \frac{B-A}{2A} \int_o^{T_n} [\dot{\sigma}^2(t) + k^2\ \sigma^2(t)]dt \geq 0. \qquad (3.6)$$

Since

$$|\dot{\sigma}(t)\ \phi_2(\sigma,t)| \leq \frac{B-A}{A} |\dot{\sigma}(t)\ k\ \sigma(t)|$$

$$\leq \frac{.5(B-A)}{A} [\dot{\sigma}^2(t) + k^2\ \sigma^2(t)]\ , \qquad (3.7)$$

(3.6) holds. (1.43) in the proof of theorem 1.2 is replaced by

$$\int_o^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt + \alpha A \int_o^{T_n} \dot{\sigma}^n(t)\ \phi_m{}^n(t)dt$$

$$+ \alpha \int_o^{T_n} \dot{\sigma}\ (t)\phi_2(\sigma,t)dt + .5\alpha \frac{(B-A)}{A} \int_o^{T_n} ([\dot{\sigma}^n(t)]^2 + [k^2\ \sigma^n(t)]^2)dt$$

$$+ E \int_o^{T_n} (\sigma^n(t) - \phi^n(t)/k)\ \phi^n(t)dt =$$

$$\int_o^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma_\phi{}^n(t))\ \phi^n(t)dt + \alpha \int_o^{T_n} \dot{\sigma}_\phi{}^n(t)\ \phi^n(t)dt$$

$$+ .5\alpha \frac{(B-A)}{A} \int_o^{T_n} [\dot{\sigma}_\phi{}^n(t)]^2dt + .5\alpha \frac{(B-A)}{A} k^2 \int_o^{T_n} [\sigma_\phi{}^n(t)]^2dt$$

$$+ E \int_o^{T_n} (\sigma_\phi{}^n(t) - \phi^n(t)/k)\ \phi^n(t)dt +$$

$$\int_o^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma_i{}^n(t))\ \phi^n(t)dt + \alpha \int_o^{T_n} \dot{\sigma}_i{}^n(t)\ \phi^n(t)dt$$

$$+ \frac{\alpha(B-A)}{A} \int_o^{T_n} \dot{\sigma}_i{}^n(t)\ \dot{\sigma}_\phi{}^n(t)dt + \frac{\alpha(B-A)}{A} k^2 \int_o^{T_n} \sigma_i{}^n(t)\sigma_\phi{}^n(t)dt + E \int_o^{T_n} \dot{\sigma}_i{}^n(t)\phi^n(t)dt$$

$$+ \frac{.5\alpha(B-A)}{A} \int_o^{T_n} [\dot{\sigma}_i{}^n(t)]^2dt + \frac{.5\alpha(B-A)}{A} k^2 \int_o^{T_n} [\sigma_i^n(t)]^2dt\ . \qquad (3.8)$$

Before applying Parseval's theorem to the integrals on the right hand side of equation (3.8), the $\sigma_\phi^n(t)$ and $\dot{\sigma}_\phi^n(t)$ terms must be replaced by $\sigma_\phi^{n*}(t)$ and $\dot{\sigma}_\phi^{n*}(t)$, respectively and the upper limits on the integrals changed to ∞. The only new step required is by the $\int_0^{T_n} [\dot{\sigma}_\phi^n(t)]^2 dt$ and $\int_0^{T_n} [\sigma_\phi^n(t)]^2 dt$ terms.

Let $\dot{\sigma}_\phi^d(t) = \dot{\sigma}_\phi^{n*}(t) - \dot{\sigma}_\phi^n(t)$ and $\sigma_\phi^d = \sigma_\phi^{n*}(t) - \sigma_\phi^n(t)$ where $\dot{\sigma}_\phi^d(t)$ is that component of $\dot{\sigma}_\phi^{n*}(t)$ outside $(0,T_n)$ and $\sigma_\phi^d(t)$ is that component of $\sigma_\phi^{n*}(t)$ outside $(0,T_n)$. Then

$$\int_0^\infty [\dot{\sigma}_\phi^n(t)]^2\, dt = \int_0^\infty [\dot{\sigma}_\phi^{n*}(t)]^2 dt - \int_0^\infty [\dot{\sigma}_\phi^d(t)]^2 dt \tag{3.9}$$

and

$$\int_0^\infty [\sigma_\phi^n(t)]^2 dt = \int_0^\infty [\sigma_\phi^{n*}(t)]^2 dt - \int_0^\infty [\sigma_\phi^d(t)]^2 dt\ . \tag{3.10}$$

Using the convolution theorem together with straightforward bounding techniques gives

$$\int_0^\infty [\dot{\sigma}_\phi^d(t)]^2 dt \leq |\phi^n(t)|^2_{max} \int_{T_n}^\infty \left[\int_{t-T_n}^t |F^{-1}(j\omega G(j\omega))| d\lambda \right]^2 dt \tag{3.11}$$

and

$$\int_0^\infty [\sigma_\phi^d(t)]^2 dt \leq |\phi^n(t)|^2_{max} \int_{T_n}^\infty \left[\int_{t-T_n}^t |F^{-1}(G(j\omega))|\ d\lambda \right]^2 dt. \tag{3.12}$$

With (3.9) and (3.10) used on the modified right hand side of (3.8), the Aizerman and Gantmacher completing the square approach together with condition (3.1) gives a bound on all of the integrals on the modified right hand side except for

$$\int_0^\infty [x(t) * \sigma_\phi{}^d(t)]\phi^n(t)dt + \frac{.5\alpha(B-A)}{A}\int_0^\infty [\dot{\sigma}_\phi{}^d(t)]^2 dt +$$

$$\frac{.5\ (B-A)}{A} k^2 \int_0^\infty [\sigma_\phi{}^d(t)]^2 dt + \frac{.5\alpha(B-A)}{A}\int_0^{T_n} [\dot{\sigma}_i{}^n(t)]^2 dt +$$

$$\frac{.5\alpha(B-A)}{A}\int_0^{T_n} [\sigma_i{}^n(t)]^2 dt\ . \tag{3.13}$$

Using the result obtained for the first integral of (3.13) in Chapter I together with (3.11) and (3.12) gives that the left hand side of (3.8) is less than or equal to

$$M_1 + |\phi^n(t)|^2_{max} M_2 \tag{3.14}$$

where M_1 and M_2 are positive numbers independent of T_n. Using (3.6) and (3.7) gives

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt + \alpha A\ \Phi_m(t)$$
$$\leq \alpha A\ \Phi_m(0) + M_1 + |\phi^n(t)|^2_{max} M_2 \tag{3.15}$$

where $\Phi_m(t) = \int_0^{\sigma(t)} \phi_m(\sigma)d\sigma$. The above inequality shows that $\sigma(t)$ is bounded and that $\int_0^{T_n} \sigma(t)\phi(t)dt$ is also bounded, thereby demonstrating asymptotic stability in the large. Q.E.D.

Example 3.1. Let $G(s) = \frac{(s + .0001)(s + .05)}{(s + .1)(s + 1)^3}$. The problem is to find the characteristics of the time varying nonlinearity that is permitted if the system is to be asymptotically stable in the large. $G(j\omega)$ has a leading phase angle outside the $\pm 90^{\circ}$ band at low frequencies and a lagging angle outside this band at high frequencies. A convenient choice for $Z(s)$ is $(-s+.05)(s+1)/(-s+.1)$. $Z(s)\ G(s)$ is then $(-s^2 + .0025)(s + .0001)/(-s^2 + .01)(s + 1)^2$, the real part of which is non-negative for all ω. Also, since $Z(s) = s + 1.05 - .055/(-s + .1)$, both $x(t)$ and $y(t)$ are non-positive and $\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)dt = .524$. Therefore, from (3.3) it follows that $B/A < 1.91$. Next k is determined by working with (3.1) with $E = 0$. The largest allowed value is $k = 2.10$. Therefore, any continuous time varying nonlinearity with a monotone bounding function $\phi_m(\sigma)$ such that the B/A inequality is satisfied and having a linear bound with a slope less than 2.10 is permitted. An example of an allowed function is $\phi(\sigma,t) = p\sigma(1 + q \cos \omega_o t)/(1 + |\sigma|)$, where $0 < p < 1.6$ and $0 < q < .312$. For this case $\phi_m(\sigma) = p\sigma/(1 + |\sigma|)$.

The next theorem gives a bound on the response for ϕ being a time varying nonlinearity.

<u>Theorem 3.2</u>. For the system of figure 1 excited by initial conditions let a and b of theorem 3.1 hold and let

c. $$\text{Re } H(j\omega) = \text{Re}[c(1 + dj\omega + X(j\omega) + Y(j\omega))\, G(j\omega - a) + E(G(j\omega - a) + 1/k) + da\, G(j\omega - a) + d\,\frac{(B-A)}{2A}(k^2 + a^2 + \omega^2)\,|G(j\omega - a)|^2] \geq \delta > 0 \tag{3.16}$$

for all real ω where a is a positive number whose magnitude is less than the magnitude of the real part of the pole of G(s) closest to the $j\omega$ axis and c, d and E are positive numbers. x(t) and y(t) are composed of delayed impulses and a piecewise continuous function that satisfies $x(t)=0$ for $t > 0$, $y(t) = 0$ for $t < 0$ and $x(t) \leq 0$ for $t < 0$ and $y(t) \leq 0$ for $t > 0$. The magnitude of the piecewise continuous component of x(t) is assumed to be less than $\ell \exp(ft)$ where ℓ and f are positive numbers and

$$\int_{-\infty}^{+\infty} e^{-a|t|}\,|x(t) + y(t)|\,dt < 1. \tag{3.17}$$

Then, $$\Phi_m(T_n) \leq e^{-2aT_n}\left[\frac{\int_0^\infty m^2(t)\,dt}{4dA} + \Phi_m(0) + \frac{M(T_n)\,|\phi^n(t)|^2_{max}}{dA} + \frac{R(T_n)}{dA}\right] \tag{3.18}$$

where $\Phi_m(T_n) = \int_0^{\sigma(T_n)} \phi_m(\sigma)\,d\sigma$ and $m(t) = F^{-1}[P(j\omega)\,Q(j\omega)]$ with

$$p(t) = e^{at}\,[(c + 2ad + E)\,\sigma_i^{\,n}(t) + d\,\dot{\sigma}_i^{\,n}(t)] + c\,[\sigma_i^{\,n}(t)e^{at} * (x(t) + y(t))]^n + \frac{d(B-A)}{A}\,e^{at}\,[\sigma_i^{\,n}(t) + k^2\sigma_i^{\,n}(t)].$$

$Q(j\omega)$ is defined by $1/\mathrm{Re}\, H(j\omega) = Q(j\omega)\, Q(-j\omega)$,

$$M(T_n) = c \int_0^{T_n} e^{at} \int_{-\infty}^{t} e^{a(t-\lambda)} \int_{t-\lambda-T_n}^{t-\lambda} |g(\varepsilon)||x(\lambda)|\, d\varepsilon\, d\lambda\, dt$$

$$+ \frac{.5d(B-A)}{A} \int_{T_n}^{\infty} e^{2at} \left[\int_{t-T_n}^{t} |F^{-1}(j\omega\, G(j\omega))|\, d\lambda \right]^2 dt +$$

$$\frac{.5d(B-A)}{A} \int_{T_n}^{\infty} e^{2at} \left[\int_{t-T_n}^{t} |F^{-1}(G(j\omega)|\, d\lambda \right]^2 dt, \text{ and}$$

$$R(T_n) = \frac{.5d(B-A)}{A} \int_0^{T_n} e^{2at} \left([\sigma_i^{\,n}(t)]^2 + [\dot{\sigma}_i^{\,n}(t)]^2\right) dt.$$

Proof. The proof of this theorem is similar to that of theorems 1,2, 2.4, and 3.1. A modification required for this case occurs for the $\int_0^{T_n} e^{2at}\, \dot{\sigma}(t)\, \phi(\sigma,t)dt$ term. It may be rewritten as

$$\int_0^{T_n} e^{2at}\, \dot{\sigma}(t)\, \phi(\sigma,t)dt = A \int_0^{T_n} e^{2at}\, \dot{\sigma}(t)\, \phi_m(\sigma(t))dt +$$

$$\int_0^{T_n} e^{2at}\, \dot{\sigma}(t)\, \phi_2(\sigma,t)dt. \qquad (3.19)$$

Integration by parts gives for the first integral on the right hand side of (3.19)

$$Ae^{2aT_n}\, \Phi_m(\sigma) - A\, \Phi_m(\sigma(0)) - 2aA \int_0^{T_n} e^{2at}\, \Phi_m(t)dt. \qquad (3.20)$$

The second integral on the right hand side of (3.19) is less than

$$\frac{.5(B-A)}{A} \int_{o}^{T_n} [\dot{\sigma}^2(t) + k^2 \sigma^2(t)] e^{2at} \, dt \tag{3.21}$$

Using these modifications together with the approaches already employed gives the proof of the theorem. Q.E.D.

The conditions of the theorems for the time varying case are a good deal more complicated than their time invariant counterparts; there appears to be no way of simplifying these results and still obtaining improved conditions for asymptotic stability.

B. Application to Sampled Data Systems

In this section the techniques of the foregoing work are used to derive an improved stability criterion for sampled data systems. To the authors' knowledge, the best results obtained thus far for the single nonlinearity system are due to Jury and Lee[20]. Their criterion includes that of Tsypkin[21] as a special case. For asymptotic stability in the large it is required that the following relationship be satisfied on the unit circle:

$$\mathrm{Re}\, G^*(z)\ [1 + q(z - 1)] + 1/K - K' \frac{|q|}{2} |(z - 1)G^*(z)|^2 \geq 0,$$

where $0 < \phi(\sigma)/\sigma < K$ and $|\frac{d\phi(\sigma)}{d\sigma}| < K'$. In the above inequality $(z - 1)$ is analogous to the $j\omega$ term in the Popov criterion. Theorem 3.3 given below permits an entire class of multipliers to be used.

a. A Theorem for Monotone Nonlinearities

Theorem 3.3. For the system shown in Figure 11 let the following hold:

a. $0 \leq d\phi(\sigma)/d\sigma \leq k_2$ where k_2 is a positive number, both $\phi(\sigma)$ and $\sigma - \phi(\sigma)/k_2 = 0$ only for $\sigma = \phi(\sigma) = 0$, and $d\phi(\sigma)/d\sigma$ is a continuous function of σ.

b. $G^*(z)$ is a rational function of z having all of its poles inside the unit circle and the corresponding time function $g(i)$ is zero for i negative. The numerator and the denominator of $G^*(z)$ are assumed to have no common factors outside or on the unit circle in the z plane.

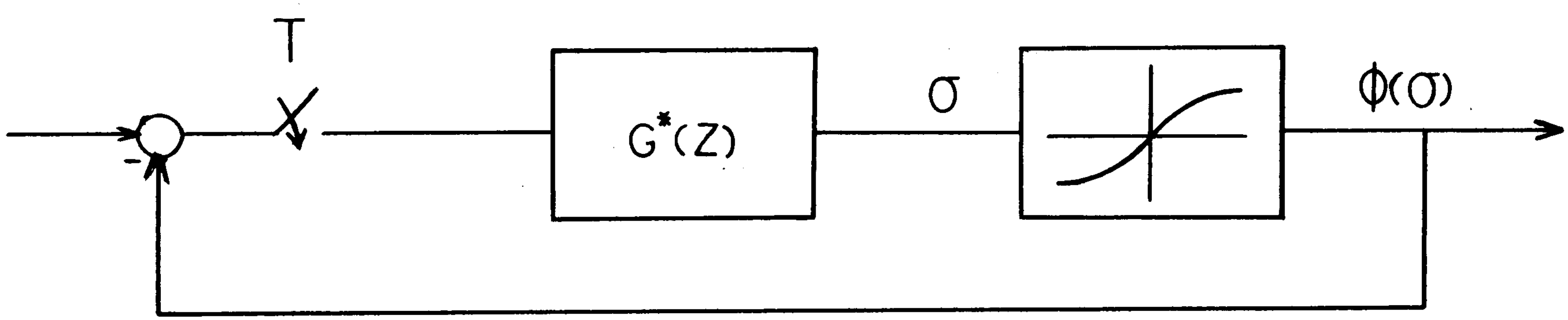

Figure 11. The Sampled Data System Under Consideration.

c. $\lim\limits_{|\sigma| \to \infty} (\sigma - \phi(\sigma)/k_2)\ \phi/|\phi(\sigma)|^2 = \infty.$

Then a sufficient condition for asymptotic stability in the large is that

$$\mathrm{Re}\ [R^*(z)\ (G^*(z) + 1/k_2)] \geq 0 \tag{3.22}$$

for $z = e^{j\omega T}$ for $0 \leq \omega \leq 2\pi$ where

$$R^*(z) = 1 + X^*(z) + Y^*(z)\ . \tag{3.23}$$

The time function $x(i) = 0$ for $i > 0$ and ≤ 0 for $i < 0$ while $y(i) = 0$ for $i < 0$ and ≤ 0 for $i > 0$. These functions must also satisfy

$$\sum_{i=-\infty}^{+\infty} (|x(i)| + |y(i)|) < 1. \tag{3.24}$$

The magnitude of $x(i)$ is less than ℓ exp (fi) where ℓ and f are positive numbers.

Corollary 1. In addition to the conditions of theorem 3.3, if $\phi(\sigma)$ is an odd monotone nonlinearity, that is, if $\phi(\sigma) = -\phi(-\sigma)$, the assertion of the theorem holds with $x(i)$ and $y(i)$ permitted to take on positive as well as negative values.

Corollary 2. If $G^*(z)$ has poles on the unit circle, $G^*(z)$ is required to be stable in the limit; that is for an arbitrarily small positive number ε, the roots of $1 + \varepsilon G^*(z)$ must all lie inside the unit circle. Also, the slope condition becomes $\geq \delta > 0$ where δ is an arbitrarily small positive number. The other conditions are unchanged except for (3.22) being $\geq \delta_1 > 0$.

Proof. First it will be shown that

$$\sum_{i=o}^{n} \phi^n(i)\ (\sigma^n(i) - \phi^n(i)/k_2) +$$

$$\sum_{i=o}^{n} \phi^n(i) \sum_{h=-\infty}^{i} [x(h) + y(h)][\sigma^n(i - h) - \phi^n(i - h)/k_2] =$$

$$c(n) \sum_{i=o}^{n} \phi^n(i)\ (\sigma^n(i) - \phi^n(i)/k_2) \tag{3.25}$$

where c(n) is a positive number. The second summation on the left hand side can be rewritten as

$$\sum_{i=o}^{n} \phi^n(i) \sum_{h=-\infty}^{o} x(h)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2] +$$

$$\sum_{i=o}^{n} \phi^n(i) \sum_{h=o}^{\infty} y(h)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2]. \tag{3.26}$$

Interchanging the order of summation gives

$$\sum_{h=-\infty}^{o} x(h) \sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2] +$$

$$\sum_{h=o}^{\infty} y(h) \sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2]\ . \tag{3.27}$$

Rewriting $\sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2]$ in terms of the positive and negative components of $\phi^n(i)$ and $\sigma^n(i - h) - \phi^n(i - h)/k_2$ and applying lemma 1 given at the end of Chapter 2 results in

$$\sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i-h) - \phi^n(i-h)/k_2] \leq$$

$$\sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i) - \phi^n(i)/k_2\ ] \tag{3.28}$$

Using (3.22) from the statement of the theorem together with (3.28) shows that (3.25) holds.

Letting $\sigma^n(i) = \sigma_\phi{}^n(i) + \sigma_i{}^n(i)$ in the left hand side of (3.25) gives for this side of the equation

$$\sum_{i=o}^{n} \phi^n(i)(\sigma_\phi{}^n(i) - \phi^n(i)/k_2) +$$

$$\sum_{i=o}^{n} \phi^n(i) \sum_{h=-\infty}^{i} [x(h) + y(h)][\sigma_\phi{}^n(i-h) - \phi^n(i-h)/k_2] +$$

$$\sum_{i=o}^{n} \phi^n(i)\ [\sigma_i{}^n(i) + \sum_{h=-\infty}^{i} [x(h) + y(h)]\ \sigma_i{}^n(i-h)] \tag{3.29}$$

Let $\sigma_\phi{}^n(i)$ be replaced by $\sigma_\phi{}^{n*}(i)$ where

$$\sigma_\phi{}^{n*}(i) = -Z^{-1}[G^*(z)\ Z[\phi^n(i)]].$$

This substitution can be made without changing the values of the summations in the first two summations of (3.29) except for the term involving x(i). Since x(i) is not zero for i < 0, the value of $\sigma_\phi{}^{n*}(i)$ for i > n will contribute to the result obtained by convolution. Therefore, the summation involving $x(i)$ is handled separately by making the substitution

$$\sigma_\phi^n(i) = \sigma_\phi^{n*}(i) - \sigma_\phi^d(i)$$

which gives for the total summation where the limits on i have been extended to $\pm\infty$,

$$\sum_{i=-\infty}^{+\infty} \phi^n(i)\ (\sigma_\phi^{n*}(i) - \phi^n(i)/k_2) +$$

$$\sum_{i=-\infty}^{+\infty} \phi^n(i) \sum_{h=-\infty}^{i} [x(h) + y(h)][\sigma_\phi^{n*}(i-h) - \phi^n(i-h)/k_2]$$

$$- \sum_{i=-\infty}^{+\infty} \phi^n(i) \sum_{h=-\infty}^{i} x(h)\ \sigma_\phi^d(i-h)\ . \tag{3.30}$$

With $\sigma_\phi^d(i) = \sum_{m=i-n}^{i} g(i-m)\ \phi^n(m),\ i \geq n$, using the exponential character of $g(i)$ and $x(i)$ as in the proof of theorem 1.1 it can be shown

$$\left| \sum_{i=-\infty}^{+\infty} \phi^n(i) \sum_{h=-\infty}^{i} x(h)\ \sigma_\phi^d(i-h) \right| \leq M_1 |\phi^n(i)|^2_{max} \tag{3.31}$$

where M_1 is a positive number independent of n and $|\phi^n(i)|_{max}$ is the largest magnitude of $\phi^n(i)$ for $0 \leq i \leq n$. Applying the Liapunov-Parseval theorem to the first two summations of (3.30) gives

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} [1 + X^*(e^{j\omega T}) + Y^*(e^{j\omega T})][G^*(e^{j\omega T}) + 1/k_2]\ |Z[\phi^n(i)]|^2\ d\omega T \tag{3.32}$$

where T is the sampling period. Since the imaginary part of the integrand does not contribute to the final result, (3.32) may be rewritten as

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathrm{Re}[1 + X^*(e^{j\omega T}) + Y^*(e^{j\omega T})][G^*(e^{j\omega T}) + 1/k_2] \; |Z[\phi^n(i)]|^2 \, d\omega T \tag{3.33}$$

which is non-positive by (3.22). Combining (3.25) with (3.29), (3.31), and (3.33) gives

$$c(n) \sum_{i=o}^{n} \phi^n(i)(\sigma^n(i) - \phi^n(i)/k_2) \leq M_1 |\phi^n(i)|^2_{max} +$$

$$\left| \sum_{i=o}^{n} \phi^n(i)[\sigma_i^{\,n}(i) + \sum_{h=-\infty}^{i} [x(h) + y(h)] \, \sigma_i^{\,n}(i-h)] \right|. \tag{3.34}$$

The second summation on the right hand side of (3.34) is less than or equal to

$$|\phi^n(i)|_{max} \sum_{i=o}^{\infty} \left| \sigma_i^{\,n}(i) + \sum_{h=-\infty}^{i} [x(h) + y(h)] \, \sigma_i^{\,n}(i-h) \right|$$

$$= |\phi^n(i)|_{max} \, M_2 \tag{3.35}$$

where M_2 is a positive number independent of n. Using (3.35) in (3.34) gives

$$c(n) \sum_{i=o}^{n} \phi^n(i)(\sigma^n(i) - \phi^n(i)/k_2) \leq M_1 |\phi^n(i)|^2_{max} + M_2 |\phi^n(i)|_{max}. \tag{3.36}$$

Let n be chosen such that $|\phi^n(i)|_{max}$ occurs at i = n. Using condition c of the statement of the theorem it follows that $\sigma^n(i)$ and $\phi^n(i)$ are bounded. Also, since the right hand side of (3.36) is independent of n, it follows that $\sigma^n(i)$ and $\phi^n(i)$ approach zero as i approaches infinity. Because of the assumptions on $G^*(z)$, it also follows that the other state variables of the system are also bounded and approach zero as $i \to \infty$. Therefore, the system is asymptotically stable in the large. Q.E.D.

The assertion of corollary 1 follows from the application of the lemma given at the end of chapter 2 to $|\sum_{i=o}^{n} \phi^n(i)(\sigma^n(i-h) - \phi^n(i-h)/k_2)|$ to get as a bound on this quantity $\sum_{i=o}^{n} \phi^n(i)(\sigma^n(i) - \phi^n(i)/k_2)$. The remainder of the proof is unchanged.

Corollary 2 follows from the transformation $\phi(\sigma) = \phi_1(\sigma) + \varepsilon\sigma$ and $G_1^*(z) = G^*(z)/1 + \varepsilon G^*(z)$ which results in a system that satisfies the conditions of the theorem.

An Allowed $R^*(z)$

Consider $\pi_i\left(\frac{z - b_i}{z - a_i}\right) \; \pi_h\left(\frac{z - c_h}{z - d_h}\right)$ with

$$0 < a_1 < b_1 < a_2 < b_2 \; \ldots < a_n < b_n < 1$$

$$1 < c_1 < d_1 < c_2 < d_2 \; \ldots < c_m < d_m.$$

Expansion of this function in a partial fraction expansion gives where A_i and B_h are positive numbers

$$1 - \sum_i \frac{A_i}{z-a_i} + \sum_h \frac{B_h}{z-d_h}$$

$$= 1 - \sum_i \frac{A_i z^{-1}}{1 - a_i z^{-1}} - \sum_h \frac{B_h/d_h}{1 - z/d_h}$$

from which it is seen that both x(i) and y(i) are non-positive. The total area

$$\sum_{i=-\infty}^{\infty} |x(i)| + |y(i)| = 1 - \pi_i\left(\frac{1-b_i}{1-a_i}\right) \pi_h \left(\frac{1-c_h}{1-d_h}\right) < 1.$$

Therefore, this function is an allowed one for the general monotone nonlinearity.

<u>Example 3.2</u>. Let $G^*(z) = \frac{3.6}{z-.9} - \frac{1.2}{z-.3}$ and $0 < k_2 < 1$.

$$G^*(z) + 1/k_2 = \left(\frac{z+.3}{z-.3}\right)\left(\frac{z+.9}{z-.9}\right). \quad \text{Let } R^*(z) = \frac{z-.3}{z+.3}.$$

Expressing this function in the time domain gives

$$R^*(z) = 1 - .6z^{-1} + 2(.3)^2 z^{-2} - 2(.3)^3 z^{-3} + \ldots$$

from which it is seen that y(t) takes on both positive and negative values and that the summation of the magnitude is 6/7. Therefore, this $R^*(z)$ may be used with symmetrical monotone nonlinearities.

$R^*(z)\ (G^*(z) + 1) = (z + .9)/(z - .9)$. The angle of this product on the unit circle is $-\tan^{-1}(9.48 \sin \omega T)$. Therefore, the criterion is satisfied and the system is asymptotically stable in the large for the given range of k_2.

C. The Multiple Nonlinearity Problem

Application of the by now standard approach gives the following theorem for a system having a number of nonlinearities.

Theorem 3.4.

For a continuous system with i nonlinearities let the following conditions hold:

a. $0 \leq d\,\phi_i(\sigma_i)/d\sigma_i \leq k_{2i}$ where k_{2i} is a positive number, both $\phi_i(\sigma_i)$ and $\sigma_i - \phi_i(\sigma_i)/k_{2i} = 0$ only for $\sigma_i = \phi_i(\sigma_i) = 0$, and $d\phi_i(\sigma_i)/d\sigma_i$ is a continuous function of σ_i.

b. The transfer function $-G_{ij}(s)$ relating $F(\sigma_i(t))$ to $F(\phi_j(t))$ is a rational function of s with the number of zeros at least one less than the number of poles and with all of the poles in the left half s plane.

c. $$\lim_{|\sigma_i| \to \infty} \int_0^{\sigma_i} \phi_i(\sigma_i)d\sigma_i/|\phi_i(\sigma_i)|^2 = \infty.$$

Then a sufficient condition for asymptotic stability in the large is that the Hermitian matrix $H(j\omega)$ be positive semi-definite where

$$H(j\omega) = \begin{bmatrix} h_{11}(j\omega) & h_{12}(j\omega) & \cdots\cdots \\ h_{21}(j\omega) & h_{22}(j\omega) & \cdots\cdots \\ & & \\ & & h_{nn}(j\omega) \end{bmatrix}$$

where $h_{ii}(j\omega) = \text{Re}\, Z_i(j\omega)\, [(G_{ii}(j\omega) + 1/k_{2i})]$ and

$$h_{ij}(j\omega) = \frac{1}{2}\,[Z_i(j\omega)\; G_{ij}(j\omega) + \overline{Z_j(j\omega)\; G_{ji}(j\omega)}] \text{ for } i < j$$

and $\quad h_{ij}(j\omega) = \overline{h_{ji}(j\omega)}$ for $i > j$.

$$Z_i(j\omega) = 1 + \alpha_i j\omega + X_i(j\omega) + Y_i(j\omega)$$

where α_i is a positive number, $x_i(t) = 0$ for $t > 0$ and $y_i(t) = 0$ for $t < 0$ with both of these functions being non-positive and consisting of the sum of a piecewise continuous function which is Fourier transformable and shifted impulse functions that satisfy

$$\int_{-\infty}^{+\infty} (|x_i(t)| + |y_i(t)|)dt < 1.$$

<u>Corollary 1</u>. In addition to the conditions of theorem 1, if $\phi_i(\sigma_i)$ is an odd monotone nonlinearity, the assertion of the theorem holds with $x_i(t)$ and $y_i(t)$ being permitted to take on positive as well as negative values.

<u>Proof</u>. The proof of this theorem parallels that of theorem 1.1 but instead of working with one function there are n functions. The only variation occurs after applying Parseval's theorem. The quadratic form that is obtained is associated with a Hermitian matrix which is required to be positive definite. After applying this condition, the inequality given below is obtained.

$$\sum_{i=1}^{n} c_i\,(T_n)\int_o^{T_n}[\sigma_i^n(t)-\phi_i^n(t)/k_2]\,\phi_i^n(t)dt\;+$$

$$\sum_{i=1}^{n}\alpha_i\;\Phi_i(T_n)\;\le\;\sum_{i=1}^{n} M_{1i}\,|\phi_i{}^n(t)|^2_{max}+\sum_{i=1}^{n} M_{2i}\,|\phi_i{}^n(t)|_{max}$$

$$+\sum_{i=1}^{n}\alpha_i\;\Phi_i(0).$$

The reasoning of theorem 1 leads to the conclusion that all of these variables are bounded and approach zero as $t \to \infty$.

Example 3.3. This example was considered by Ibrahim and Rekasius[22]. The system consists of two nonlinearities connected in a single loop with linear elements in between. $G_1(s) = 1/(s+5)$ and $G_2(s) = (s+1)/(s+2)(s+3)$. For this case, $G_{11}(s) = G_{22}(s) = 0$, $G_{12}(s) = -1/(s+5)$ and $G_{21}(s) = (s+1)/(s+2)(s+3)$. The + sign for $G_{21}(s)$ is due to the feedback being negative. It is assumed that both nonlinearities are continuous monotone functions.

$$|H(j\omega)| = \frac{ReZ_1}{k_{21}}\frac{ReZ_2}{k_{22}} - \frac{1}{4}\left|\frac{Z_1(j\omega)}{(j\omega+5)} - \frac{Z_2(-j\omega)\qquad(-j\omega+1)}{(-j\omega+2)(-j\omega+3)}\right|^2 \ge 0.$$

If asymptotic stability in the large is to be shown for $0 < k_{21} < \infty$ and $0 < k_{22} < \infty$, two functions $Z_1(j\omega)$ and $Z_2(j\omega)$ must be found such that the quantity inside the magnitude squared brackets is zero. This requires that

$$\frac{Z_1(j\omega)}{(j\omega+5)} = \frac{Z_2(-j\omega)(-j\omega+1)}{(-j\omega+2)(-j\omega+3)}\;.$$

Let $Z_2(j\omega) = \dfrac{(-j\omega + 1)(j\omega + 3)}{(-j\omega + 2)} = j\omega + 4 - \dfrac{5}{-j\omega + 2}$ and

$$Z_1(j\omega) = \frac{(j\omega + 5)(-j\omega + 1)(j\omega + 1)}{(-j\omega + 2)(j\omega + 2)} = j\omega + 5 - \frac{\frac{9}{4}}{j\omega + 2} - \frac{\frac{21}{4}}{-j\omega + 2}.$$

A check of the integral magnitude condition for these two functions reveals that they are allowed functions for general monotone nonlinearities. Substitution of these expressions gives $(\omega^2 + 1)/(\omega^2 + 4)$ on both sides of the equation. Therefore, it has been shown that the given system is asymptotically stable in the large for monotone nonlinearities having arbitrarily large slopes. In [22], asymptotic stability was shown for $k_{21} = k_{22} = 6$.

Conclusion

This chapter has applied the method of chapters 1 and 2 to get improved theorems for a time varying nonlinearity, for a sampled data system, and for a system with a number of nonlinearities. In order to show how useful these theorems are, it will be necessary to consider a number of different examples for each case.

VI. CONCLUSION

From the conclusions given at the end of each chapter it is apparent that additional research in the area of time-frequency domain stability criteria should be worth-while. In particular, the problem of the closeness of the stability results to the actual absolute stability boundary is an important one for future study.

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III CHAPTER I. THE STABILITY OF SINGLE NONLINEARITY CONTINUOUS SYSTEMS

A. Introduction

This chapter deals mainly with sufficient conditions for the asymptotic stability in the large of the system shown in Figure 1 with $\phi(\sigma)$ a continuous monotone increasing nonlinearity. Several recent works have considered this problem [1-4]. Reference [4] by one of the authors concerns a part of the research presented in this report, namely corollary 3 of theorem 1.1 and a simplified version of theorem 1.2. Brockett and Willems [3] presented a sufficient condition for the asymptotic stability of this system with the nonlinearity being a continuous monotone function. With $0 \leq d\phi/d\sigma \leq k_2$, it was shown that

$$\mathrm{Re}[Z(j\omega) \quad (G(j\omega) + 1/k_2)] \geq 0$$

is a sufficient condition for asymptotic stability where Z is either a physically realizable RL driving point impedance function or its reciprocal. Z allows the angle of $G(j\omega) + 1/k_2$ to lie outside the $\pm 90^\circ$ band in only one direction. In other words, the polar plot of $G + 1/k_2$ is restricted to lie in three quadrants. The present work presents a theorem for the monotone nonlinearity which permits a larger class of Z multipliers to be used, thereby allowing $G + 1/k_2$ to lie in four quadrants. The same approach is applied to give improved conditions for the asymptotic stability of a system with a single odd monotone nonlinearity and for a system with a nonlinearity having a monotone bound.

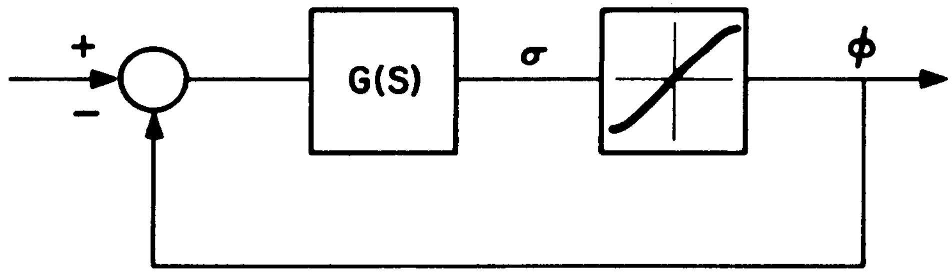

Figure 1. The continuous system under consideration.

In using the following theorems, if the nonlinear characteristic satisfies $k_1|\sigma| < |\phi(\sigma)|$, the linear transformation $\phi_1(\sigma) = \phi(\sigma) - k_1\sigma$ giving $G_1(s) = G(s)/(1 + k_1G(s))$ should first be carried out, provided that in the case of theorem 1.1 $\phi_1(\sigma)$ is a monotone increasing function. The theorems are then applied to the transformed system with nonlinear characteristic $\phi_1(\sigma)$ and transfer function $G_1(s)$.

In the following work the notation $\phi(\sigma)$ is used when the properties of the nonlinearity are under consideration and $\phi(t)$ is used when the time varying variable $\phi(\sigma(t))$ is being discussed.

B. A Theorem For Monotone Nonlinearities

Theorem 1.1

For the system shown in figure 1 let the following hold:

a. $0 \leq d\phi(\sigma)/d\sigma \leq k_2$ where k_2 is a positive number, both $\phi(\sigma)$ and $\sigma - \phi(\sigma)/k_2 = 0$ only for $\sigma = \phi(\sigma) = 0$, and $d\phi(\sigma)/d\sigma$ be a continuous function of σ.

b. $G(s) = N(s)/D(s)$ with the degree of $N(s)$ at least one less than the degree of $D(s)$ and with the zeros of $D(s)$ in the left half s plane. $N(s)$ and $D(s)$ are assumed to have no common factors in the right half s plane or on the $j\omega$ axis.

c. $\lim\limits_{|\sigma| \to \infty} \int_0^\sigma \phi(\sigma)d\sigma/|\phi(\sigma)|^2 = \infty$ or

$\lim\limits_{|\sigma| \to \infty} |\phi(\sigma)| = h|\sigma|$ where $h > 0$.

Then a sufficient condition for asymptotic stability in the large is that

$$\mathrm{Re}[Z(s)(G(s) + 1/k_2)] \geq 0 \tag{1.1}$$

for $s = j\omega$ for all real ω where

$$Z(s) = 1 + \alpha s + X(s) + Y(s). \tag{1.2}$$

The time function $x(t) = 0$ for $t > 0$ and $y(t) = 0$ for $t < 0$. Both of these functions are assumed to be the sum of a piecewise continuous function which is Fourier transformable and shifted impulse functions that satisfy

$$\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)\, dt < 1 \tag{1.3}$$

with both $x(t)$ and $y(t) \leq 0$. The magnitude of the piecewise continuous component of $x(t)$ is assumed to be less than $\ell \exp(ft)$ where ℓ and f are positive numbers. The contribution of the impulses to the integral is to be taken as the strengths of the impulses. α is a positive number.

Corollary 1. In addition to the conditions of theorem 1.1, if $\phi(\sigma)$ is an odd monotone nonlinearity that is, if $\phi(\sigma) = -\phi(-\sigma)$, the assertion of the theorem holds with (1.3) becoming

$$\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)\, dt < 1 \text{ where } x(t) \text{ and } y(t) \text{ are}$$

permitted to take on positive as well as negative values.

Corollary 2. If $G(s)$ has poles on the $j\omega$ axis, $G(s)$ is required to be stable in the limit; that is, for an arbitrarily small positive number

ε, the zeros of $1 + \varepsilon G(s)$ must all be in the left half s plane. Also, the slope condition becomes $\geq \delta > 0$ and (1.1) becomes $\geq \delta_2 > 0$ where δ and δ_2 are small positive numbers. The other conditions are unchanged.

Corollary 3. If c is not satisfied, the assertion of the theorem holds with $x(t)$ required to be identically zero.

Since the statement of the theorem is somewhat involved, a discussion of its various conditions is in order. The slope bound condition a includes a requirement that $d\phi(\sigma)/d\sigma$ be a continuous function of σ whose purpose is to insure the Fourier transformability and piecewise continuity of $\sigma(t)$, $\dot{\sigma}(t)$, and $\phi(t)$; any other property insuring this result would suffice. Condition b is used to guarantee that if $\sigma(t)$ and $\phi(t)$ are bounded for all t and approach zero as $t \to \infty$, the other state variables of the system have this same type of behavior. In addition, having the degree condition holding allows the αs term to be used in the frequency domain criterion since it insures the Fourier transformability of that component of $d\sigma(t)/dt$ due to $-\phi(t)$. The first part of condition c permits the nonlinear characteristic to have a behavior which ranges from that of a saturation function to a linear characteristic for large values of σ, with the first mentioned function being allowed but not the second. The second part of this condition permits a linear characteristic.

C. Application of the Theorem

In applying the theorem it is convenient to first draw the log magnitude and phase plots of $G(j\omega) + 1/k_2$. Since $|G(j\omega)|$ approaches zero for ω sufficiently large, above a certain frequency, ω_c, $|G(j\omega)| < 1/k_2$, and hence the phase angle of $G(j\omega) + 1/k_2$ will be less than 90^o. The real part condition will be satisfied with $Z(s) = 1$ for $\omega \geq \omega_c$. If it is also satisfied for $\omega < \omega_c$, asymptotic stability will be guaranteed.

If the real part condition is not satisfied for $\omega < \omega_c$, a frequency varying $Z(j\omega)$ must be chosen in an attempt to show stability. Since the real part condition is already satisfied for $\omega > \omega_c$, $Z(j\omega)$ should not disturb this property. The general philosophy to be followed in searching for a suitable $Z(j\omega)$ function is to observe the frequency bands in which the angle of $G(j\omega) + 1/k_2$ lies outside the $\pm 90^o$ band and then to try to choose a $Z(j\omega)$ function such that its phase angle when added to that of $G(j\omega) + 1/k_2$ gives a resultant phase angle which lies within the $\pm 90^o$ band.

D. Two Z(s) Multipliers

1. $$\prod_{i=1}^{n} \left(\frac{s + a_i}{s + b_i}\right) \prod_{j=1}^{m} \left(\frac{s - c_j}{s - d_j}\right) + \alpha s$$

$$0 < a_1 < b_1 < a_2 < b_2 < \dots a_n < b_n$$

$$0 < c_1 < d_1 < c_2 < d_2 < \dots c_m < d_m$$

The first product is an RL impedance function and the second is transformed into an RL impedance function by replacing s with -s. Therefore, the poles and zeros of the first product alternate along the negative real axis while the critical points of the second product are along the positive real axis of the s plane. Expressing this function in a partial fraction expansion gives

$$1 - \sum_{i=1}^{n} \frac{f_i}{s + b_i} - \sum_{j=1}^{m} \frac{l_j}{-s + d_j} + \alpha s$$

where f_i and l_j are positive numbers. Since the partial fraction expansion coefficients are negative for both the left half and right half plane poles, the time function corresponding to these poles is non-positive. Using $F(0) = \int_{-\infty}^{+\infty} f(t)dt$, where $F(j\omega)$ is the Fourier transform of $f(t)$, in conjunction with

$$\prod_{i=1}^{n} \left(\frac{s + a_i}{s + b_i}\right) \prod_{j=1}^{m} \left(\frac{s - c_j}{s - d_j}\right) - 1$$

gives

$$\prod_{i=1}^{n} \left(\frac{a_i}{b_i}\right) \prod_{j=1}^{m} \left(\frac{c_j}{d_j}\right) - 1 \text{ as the area associated with } x(t) + y(t)$$

for this Z(s). Since these time functions are non-positive and the magnitude of this area is less than 1, the given function is an allowed one for general monotone nonlinearities.

The phase characteristic of this function is more flexible than the Z(s) multipliers considered in [4] because it is possible to switch back and forth from a leading to a lagging function or vice versa if desired. A typical phase angle plot is shown in Figure 2 for the particular case $n = 2$, $m = 2$. It is to be noted that the magnitude of the angle can approach 90° as closely as desired.

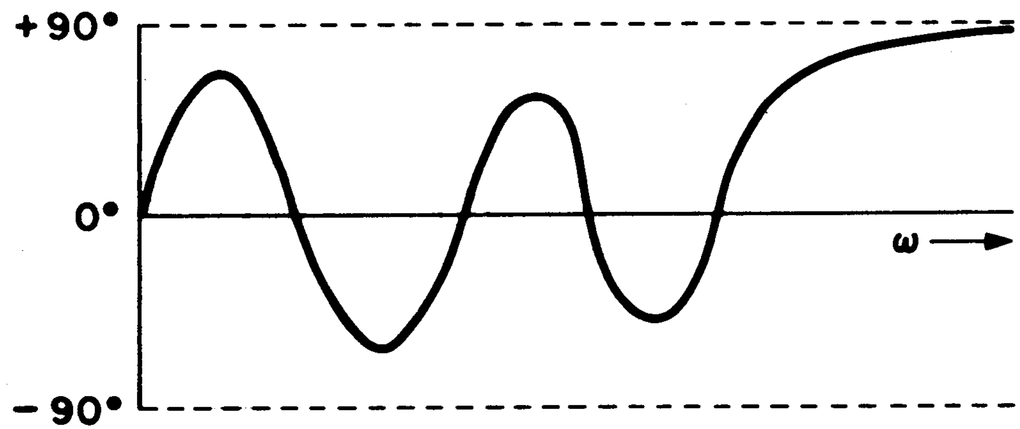

Figure 2. Angle plot for a particular type 1 function.

Example 1. Brockett and Willems [3] indicated that

$$G(s) = \frac{s^2}{s^4 + a\,s^3 + b\,s^2 + c\,s + d}$$

with a, b, c, and d chosen such that the poles of $G(s)$ lie in the left half s plane was a worthwhile function for future study since their criterion did not apply to it. This $G(s)$ is to be considered assuming that k_2 is large but finite with the nonlinearity required to be monotone. An angle plot of $G(j\omega) + 1/k_2$ is given in Figure 3. Let $Z(s) = (-s + p)(s + r)/(-s + q)$ with $p < q$. Division of the numerator by the denominator shows that this $Z(s)$ belongs to the function 1 class with $n = 0$, $m = 1$. The reason for this choice of $Z(j\omega)$ is that its angle lags at low frequency and leads at high frequency, which is the required behavior if the angle of the product function is to lie within the $\pm\, 90^{\circ}$ band. The variation in angle for $G(j\omega) + 1/k_2$ at low frequency can be handled by choosing p sufficiently small. However, a problem is encountered in following the variation from near + 180 to - 180°. First, let $G(s)$ have four real poles located at $-a_1$, $-a_2$, $-a_3$, and $-a_4$. Then $Z(s)(G(s)+1/k_2)$ is with $s = j\omega$

$$\frac{(-s + p)}{(-s + q)}\,(s + r)\;\left[\frac{s^2}{(s + a_1)(s + a_2)(s + a_3)(s + a_4)}\right] + \frac{R(s)}{k_2}$$

where $R(s)$ is the even part of $Z(s)$. The angle of the first term above with $q = a_1$ and $r = a_2$ is

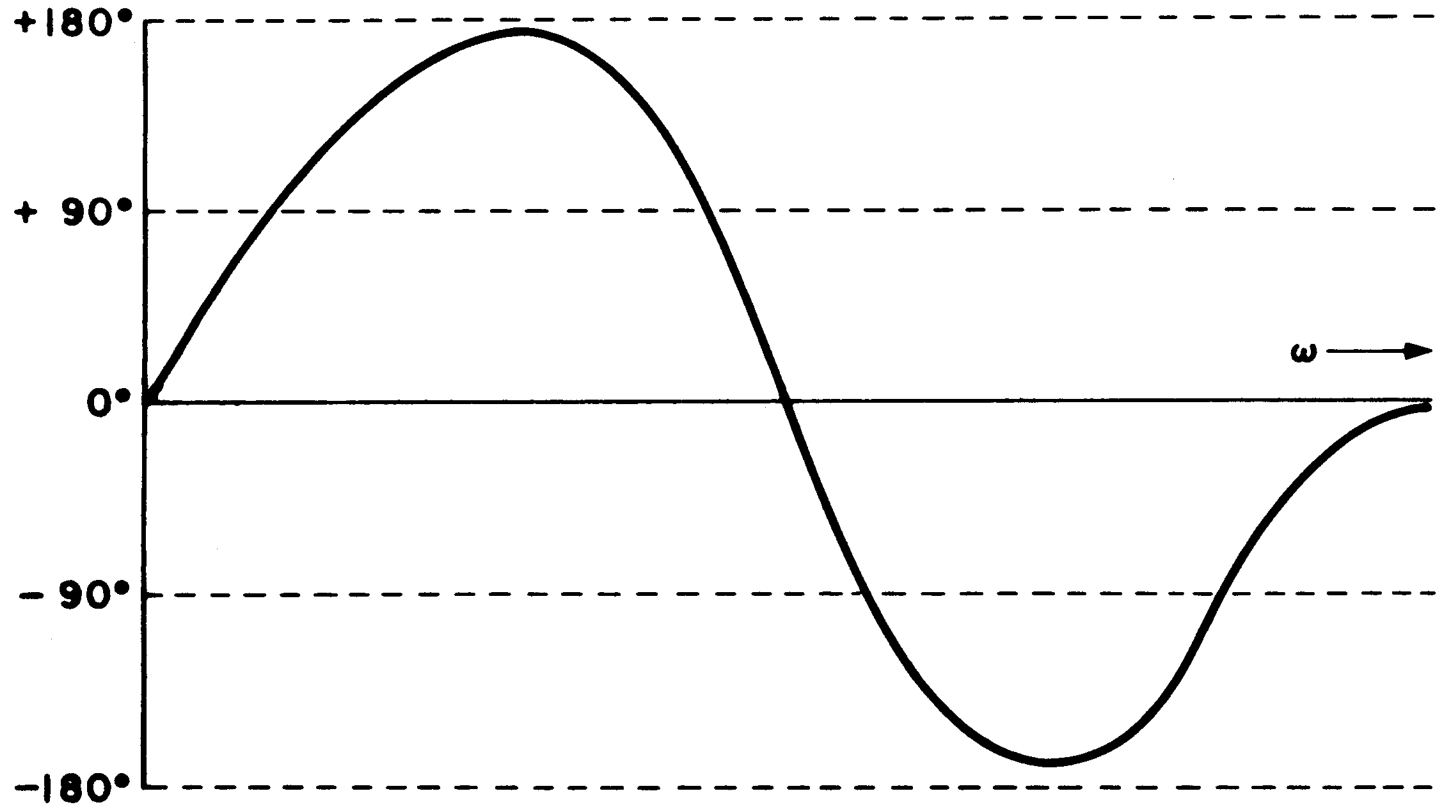

Figure 3. Angle plot of $G(j\omega) + 1/k_2$ for example 1.

$$180^\circ - \tan^{-1}\frac{\omega}{p} - \tan^{-1}\frac{\omega}{a_3} - \tan^{-1}\frac{\omega}{a_4}.$$

The value of p can be chosen small enough such that at low frequency when the magnitude of the first term is equal to $R(j\omega)/k_2$ which itself is positive, the above angle is less than 90°. Since (1.1) is satisfied, asymptotic stability in the large is guaranteed.

Next, consider the case of $G(s) = s^2/((s^2 + 2\zeta\omega_m s + \omega_m^2) \times (s + a_1)(s + a_2))$ where $\zeta < 1$ and $\omega_m > 0$. The angle of $Z(j\omega)\ G(j\omega)$ in this case with q and r chosen equal to a_1 and a_2 respectively is

$$180^\circ - \tan^{-1}\frac{\omega}{p} - \tan^{-1}\frac{2\zeta\omega_m}{(\omega_m^2 - \omega^2)}.$$

As before, a suitable choice of p will make the angle of $Z(j\omega)(G(j\omega) + 1/k_2)$ lie in the $\pm 90^\circ$ band for all ω and asymptotic stability in the large has been shown.

Finally, let $G(s) = s^2/(s^2 + 2\zeta\omega_m s + \omega_m)^2$. The angle of $Z(j\omega)\ G(j\omega)$ is, with $r = q = \omega_m$,

$$180^\circ - \tan^{-1}\frac{\omega}{p} + 2\tan^{-1}\frac{\frac{\omega}{\omega_m}\left[1 - 2\zeta - \left(\frac{\omega}{\omega_m}\right)^2\right].}{\left[1 - (1 - 2\zeta)\left(\frac{\omega}{\omega_m}\right)^2\right]}.$$

If $\zeta > .5$, and p suitably chosen, (1.1) is satisfied and asymptotic stability in the large is demonstrated. This Z(s) will not satisfy (1.1) for $\zeta < .5$ and hence no information is available on the stability of the system. Fitts [5] has shown that periodic solutions exist with $\phi(\sigma) = \sigma^3$ and $\zeta = .01$ for the 2 pair complex conjugate pole case. The author has obtained steady state oscillations with $\phi(\sigma)$ an odd saturation function for $\zeta = .045$ and with $\phi(\sigma)$ an unsymmetrical saturation nonlinearity for $\zeta = .075$.

In summary, with a monotone nonlinearity asymptotic stability in the large can be guaranteed for the given G(s) if the poles are all real, if two are real and the other two complex, or if all four are complex provided that $\zeta > .5$.

2. $1 + \sum_{i=1}^{n} a_i \exp(b_i s) + \alpha s$ with the b_i's being real numbers,

$\alpha > 0$, and $\sum_{i=1}^{n} |a_i| < 1$. If all the a_i's are negative, this multiplier can be used for a general monotone nonlinearity but if some are positive, the nonlinearity must be an odd function. The angle of this Z(s) is $\tan^{-1} ((\sum_{i=1}^{n} a_i \sin b_i\omega + \alpha\omega)/(1 + \sum_{i=1}^{n} a_i \cos b_i\omega))$. This multiplier is capable of providing a rapid change in phase shift from near -90° to $+90^\circ$, but the periodic nature of the exponential part of this function can make it a difficult one to work with. A useful special case results when $\sum_{i=1}^{n} a_i \exp(b_i j\omega) =$

$j \sum_{i=1}^{n/2} 2a_i \sin b_i\, \omega$ with $\sum_{i=1}^{n/2} 2|a_i| < 1$. The angle of $Z(j\omega)$ for this case is $\tan^{-1}(\sum_{i=1}^{n/2} 2a_i \sin b_i\, \omega + \alpha\, \omega)$ which is simpler than the general result. On the other hand, the angle variations in the function are constrained; if α is a very small number, the angle lies in a $\pm$ 45° band at low frequencies. The use of this class of multiplier is illustrated by the following example.

Example 2. Dewey and Jury [2] considered the case of $G(s)=40/s(s+1)(s^2+.8s+16)$ using their criterion for monotone nonlinearities and showed stability for nonlinearities having a slope restricted to $(\varepsilon, 1.43)$. The system is stable for linear gains in the sector $(\varepsilon, 1.76)$. Because $G(s)$ has a pole on the $j\omega$ axis, corollary 2 must be applied rather than the theorem. From the root locus plot for $1 + \varepsilon G(s)$, where ε is a small positive number, it is seen that $G(s)$ is stable in the limit. From the Figure 4 plot of the angle of $G(j\omega) + 1/1.76$, the angle lies outside the $\pm$ 90° band in the frequency ranges 0-2.75 and 2.97-3.75, lagging in the former case and leading in the latter. Although the peak deviation outside the $\pm$ 90° band is only 36° in the lagging direction and 16° in the leading direction, the peak slope of the angle in making the transition from outside the $\pm$ 90° band to the inside is about 40°/radian, making it impossible to use a $Z(s)$ of the function 1 class. The magnitude of the slope of a $Z(s)$ function belonging to the type 1 class is less than or equal to the slope of the angle of the double pole function $2 \tan^{-1} \omega/a$ which is $2a/(\omega^2 + a^2)$. For $\omega = a = 3$, approximately the values which would have to be chosen in attempting to use the function, the slope would be about 20°/radian, less than half the required value. Therefore, a function of the type 2 class is chosen in an effort to show asymptotic stability. Since the required angle for $Z(s)$ is less

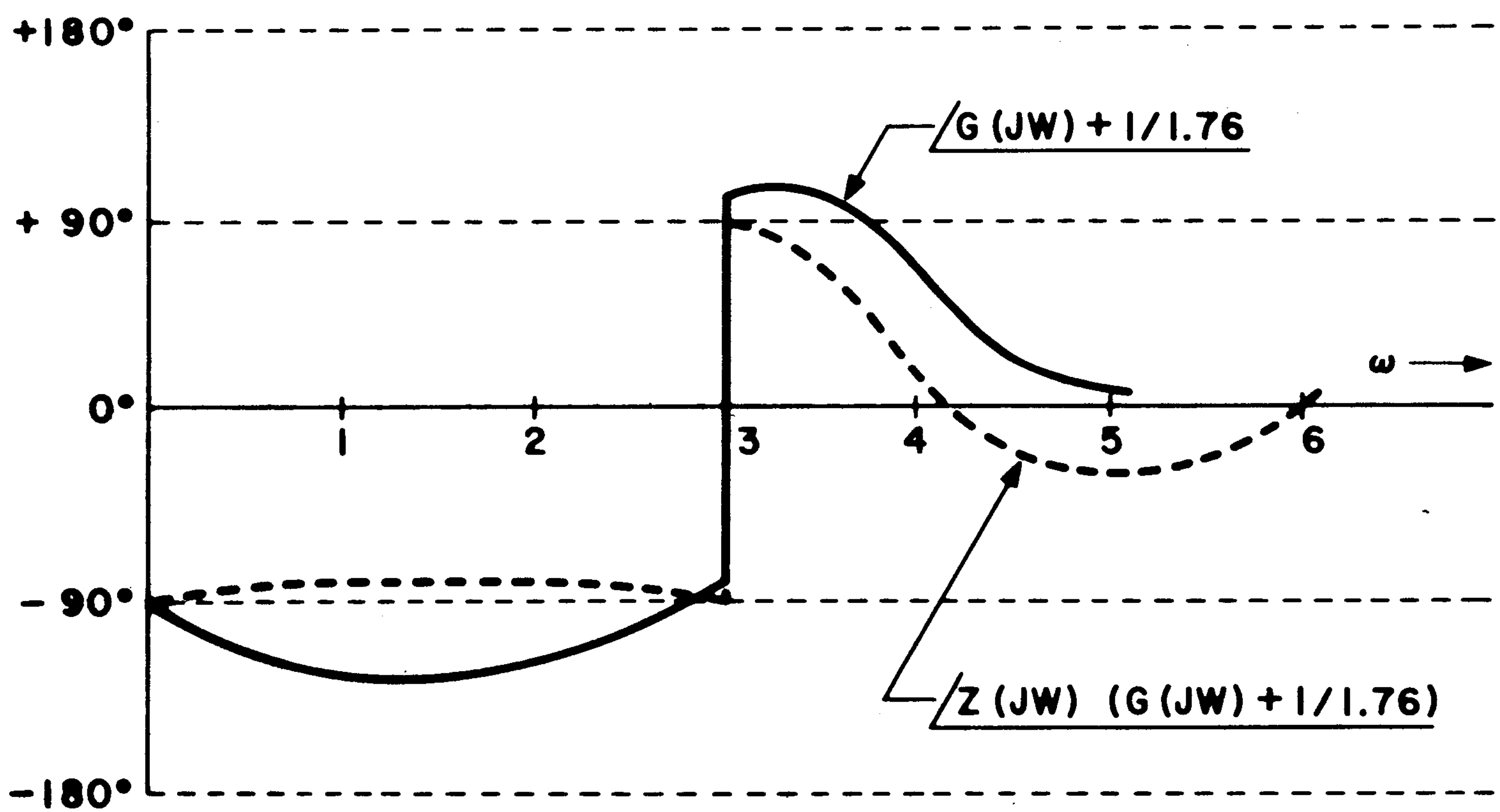

Figure 4. Angle plots pertinent to the first part of example 2.2.

than 45°, and a leading angle followed by a lagging angle is required, $Z(j\omega)$ was chosen equal to $1 + j.999 \sin 1.118\omega + 10^{-10} j\omega$. Comparing this function with the time domain condition (1.3) shows that $\phi(\sigma)$ is required to be an odd monotone function. The 1.118 coefficient was picked to give an angle for $Z(j\omega)(G(j\omega) + 1/1.76)$ at $\omega = 2.98$, the frequency at which a zero occurs on the $j\omega$ axis for $G(j\omega) + 1/1.76$, of $\pm 90^{\circ}$. The amplitude of the sine term was chosen close to 1 to give a large change in angle while still satisfying the integral condition (1.3)and the 10^{-10} coefficient was chosen so that the $\alpha\, j\omega$ term does not come into play at low frequencies. The slope of the angle of this multiplier at $\omega = 3$ is about 60°/radian. The plot of the angle of the product function also given in Figure 4 shows that the angle always remains within the $\pm 90^{\circ}$ except for $\omega = 0$, 2.98, and ∞ at which frequencies the angle magnitude is 90°. Calculation of the real part of the product function at $\omega = 0$ gives .738. If $k_2 < 1.76$, the real part of the product is positive at $\omega = 2.98$. At ∞, this quantity is 1/1.76. Therefore, since (1.1)is satisfied with an inequality sign, all the conditions of corollaries 1 and 2 are satisfied and asymptotic stability in the large is guaranteed for slopes in the sector $(\varepsilon, 1.76)$ for $\phi(\sigma)$ equal to an odd monotone nonlinearity.

In order to find an enlarged sector of assured asymptotic stability for the general monotone nonlinearity, $Z(j\omega) = 1 - .95 \exp(-1.045j\omega) + 10^{-10} j\omega$ was chosen for use with $G(j\omega) + 1/1.7$. The reasons for the choice of this function and the parameters for this case are identical with those of the previous case except that the coefficient of the exponential was chosen to give a zero phase shift for $Z(j\omega)$ in the middle of the transition region for the angle of $G(j\omega) + 1/1.7$. The slope of the angle of this $Z(j\omega)$ at $\omega = 3$ is about 30^o/radian. Therefore, k_2 was reduced to 1.7 when it was found to be impossible to satisfy (1.1) with the given form of $Z(j\omega)$ and $k_2 = 1.76$. Figure 5 gives the pertinent plots for this example which show that the angle of $Z(j\omega)(G(j\omega) + 1/1.7)$ is in the $\pm 90^o$ band. At $\omega = 0$ the angle of the product is -90^o but the real part is 2.38 while at $\omega = \infty$ the angle is 90^o with the real part being $(1-.95\cos 1.045\omega)/1.7$. Therefore, the conditions of corollary 2 are satisfied and asymptotic stability in the large is guaranteed for the general monotone nonlinearity with slope in the sector $(\varepsilon, 1.7)$.

E. Proof of Theorem 1.1

Let the system be excited by initial conditions. The assumptions on $G(s)$ and on $\phi(\sigma)$ are sufficient to insure the continuity and Fourier transformability of $\sigma(t)$, $\dot{\sigma}(t)$, and $\phi(t)$ on any finite time interval [6] - [9]. Use will be made of these properties at several points in the proof. First, it will be shown that

$$\int_o^{T_n} ((\delta(t) + x(t) + y(t)) * (\sigma^n(t) - \phi^n(t)/k_2))\phi^n(t)\, dt =$$

$$c(T_n) \int_o^{T_n} (\sigma^n(t) - \phi^n(t)/k_2)\, \phi^n(t)dt \qquad (1.4)$$

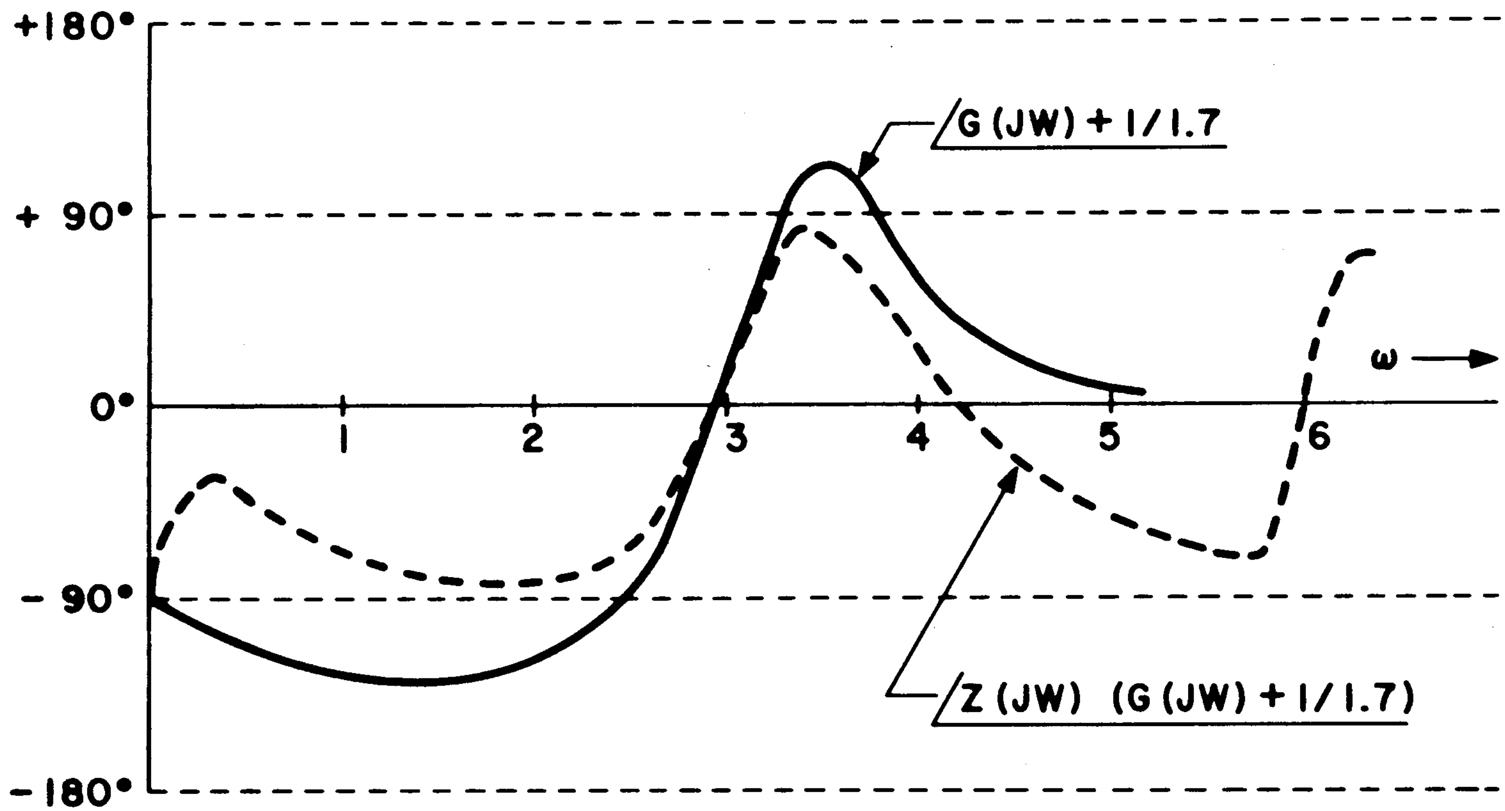

Figure 5. Angle plots for the second part of example 1.2.

where $c(T_n)$ is a positive number and * denotes convolution. The variables $\sigma^n(t)$ and $\phi^n(t)$ are equal to their non-superscripted counterparts in $(0, T_n)$ and zero outside this interval. Let $x'(t)$ and $y'(t)$ denote $x(t)$ and $y(t)$ respectively with the impulses removed. The integral involving these functions on the left hand side of (1.4) is given by

$$\int_0^{T_n} \int_{-\infty}^{0} x'(\lambda)\,(\sigma^n(t-\lambda)-\phi^n(t-\lambda)/k_2)\phi^n(\lambda)\;d\lambda\;dt\;+$$

$$\int_0^{T_n} \int_0^{\infty} y'(\lambda)\;(\sigma^n(t-\lambda)\;-\;\phi^n(t-\lambda)/k_2)\;\phi^n(t)\;d\lambda\;dt \qquad (1.5)$$

since $x'(\lambda) = 0$ for $\lambda > 0$ and $y'(\lambda) = 0$ for $\lambda < 0$. Because the primed functions, $\sigma^n(t)$ and $\phi^n(t)$ are continuous functions of t, and the integrand is non-zero over only a finite interval of time, the order of integration may be interchanged [10] to give

$$\int_{-\infty}^{0} x'(\lambda) \int_0^{T_n} (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\;\phi^n(t)\;dt\;d\lambda\;+$$

$$\int_0^{\infty} y'(\lambda) \int_0^{T_n} (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\;\phi^n(t)\;dt\;d\lambda. \qquad (1.6)$$

With the impulsive component of x(t) given by $\sum_{i=1}^{u} a_i\;\delta(t + b_i)$ and that of y(t) by $\sum_{j=1}^{v} c_i\;\delta(t - d_i)$, where a_i, b_i, c_i, and d_i are positive numbers, their contribution to the left hand side of (1.4) is

$$\sum_{i=1}^{u} a_i \int_0^{T_n} (\sigma^n(t+b_i) - \phi^n(t+b_i)/k_2)\;\phi^n(t)\;dt\;+$$

$$\sum_{j=1}^{v} c_i \int_0^{T_n} (\sigma^n(t-d_i) - \phi^n(t-d_i)/k_2)\;\phi^n(t)\;dt\;. \qquad (1.7)$$

Appearing in both (1.6) and (1.7) is an integral of the form

$$I(T) = \int_0^{T_n} (\sigma^n(t - T) - \phi^n(t - T)/k_2)\ \phi^n(t)\ dt \qquad (1.8)$$

where T is a real number. At this point a positive bound will be developed on (1.8). Let (1.8) be rewritten as

$$\begin{aligned} I(T) = & \int_0^{T_n} \left(\sigma_+^n(t - T) - \frac{\phi_+^n(t - T)}{k_2}\right) \phi_+^n(t)\ dt\ + \\ & \int_0^{T_n} \left(\sigma_-^n(t - T) - \frac{\phi_-^n(t - T)}{k_2}\right) \phi_-^n(t)\ dt\ + \\ & \int_0^{T_n} \left(\sigma_+^n(t - T) - \frac{\phi_+^n(t - T)}{k_2}\right) \phi_-^n(t)\ dt\ + \\ & \int_0^{T_n} \left(\sigma_-^n(t - T) - \frac{\phi_-^n(t - T)}{k_2}\right) \phi_+^n(t)\ dt \end{aligned} \qquad (1.9)$$

where the + and - subscripts refer to the positive and negative values of the associated functions, respectively: as an example $\phi_+^n(t)$ is equal to $\phi^n(t)$ when $\phi^n(t) > 0$ and zero otherwise. The lemma may be applied to the first two integrals since $\sigma^n(t)$ and $\phi^n(t)$ are continuous functions of time that are zero outside $(0, T_n)$ the two functions forming the integrand of both integrals are non-negative and non-positive respectively, and

$$\begin{aligned} d(\sigma-\phi(\sigma)/k_2)/d\phi(\sigma) &= [d(\sigma-\phi(\sigma)/k_2)/d\sigma][d\sigma/d\phi(\sigma)] = \\ & [1 - (d\phi(\sigma)/d\sigma)(1/k_2)][d\sigma/d\phi(\sigma)] \geq 0\ , \end{aligned} \qquad (1.10)$$

showing that $\sigma - \phi(\sigma)/k_2$ is a monotone increasing function of $\phi(\sigma)$. Applying the lemma gives

$$I(T) \leq \int_0^{T_n} (\sigma_+^n(t-T) - \frac{\phi_+^n(t-T)}{k_2}) \phi_+^n(t)\, dt + \int_0^{T_n} (\sigma_-^n(t-T) - \frac{\phi_-^n(t-T)}{k_2}) \times$$

$$\phi_-^n(t)\, dt \leq \int_0^{T_n} (\sigma_+^n(t) - \frac{\phi_+^n(t)}{k_2}) \phi_+^n(t)\, dt + \int_0^{T_n} (\sigma_-^n(t) - \frac{\phi_-^n(t)}{k_2}) \phi_-^n(t)\, dt$$

$$= \int_0^{T_n} (\sigma(t) - \frac{\phi(t)}{k_2}) \phi(t)\, dt . \qquad (1.11)$$

Using (1.6) and (1.7) gives for that part of (1.4) involving $x(t)+y(t)$

$$\int_{-\infty}^{+\infty} (x'(\lambda) + y'(\lambda)) I(\lambda)\, d\lambda + \sum_{i=1}^{u} a_i I(-b_i) + \sum_{j=1}^{v} c_i I(d_i) . \qquad (1.12)$$

Now, since $x'(\lambda)$, $y'(\lambda)$, a_i and c_i are non-positive, application of (1.11) and (1.12) yields

$$\int_0^{T_n} ((x(t) + y(t)) * (\sigma^n(t) - \phi^n(t)/k_2)) \phi^n(t)\, dt \geq$$

$$[\int_{-\infty}^{+\infty} (x'(\lambda) + y'(\lambda))\, d\lambda + \sum_{i=1}^{u} a_i + \sum_{j=1}^{v} c_i]\, I(0) . \qquad (1.13)$$

Using (1.3) from the statement of the theorem it follows that the left hand side of (1.13) is greater than $-I(0)$ and hence that the assertion of (1.4) is correct.

The next step in the proof is to apply Parseval's theorem to a part of (1.4) and to use the frequency domain condition (1.1). Let $\sigma^n(t) = \sigma_\phi{}^n(t) + \sigma_i{}^n(t)$ and $\dot\sigma^n(t) = \dot\sigma_\phi{}^n(t) + \dot\sigma_i{}^n(t)$ where $\sigma_\phi{}^n(t)$ and $\dot\sigma_\phi{}^n(t)$ are those components of $\sigma^n(t)$ and $\dot\sigma^n(t)$, respectively, due to the feedback signal $-\phi(t)$ and $\sigma_i^{\,n}(t)$ and $\dot{\sigma}_i^{\,n}(t)$ are due to the initial condition excitation of the system. Then

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * (\sigma^n(t) - \phi^n(t)/k_2))\ \phi^n(t)\ dt +$$

$$\alpha \int_0^{T_n} \dot{\sigma}^n(t)\ \phi^n(t)\ dt = \int_0^{T_n} ((\delta(t) + y(t)) * (\sigma_\phi^{\,n}(t) - \phi^n(t)/k_2))\ \phi^n(t)\ dt +$$

$$\int_0^{T_n} (x(t) * (\sigma_\phi^{\,n}(t) - \phi^n(t)/k_2))\ \phi^n(t)\ dt + \alpha \int_0^{T_n} \dot{\sigma}_\phi^{\,n}(t)\ \phi^n(t)\ dt +$$

$$+ \int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma_i^{\,n}(t))\ \phi^n(t)\ dt + \alpha \int_0^{T_n} \dot{\sigma}_i^{\,n}(t)\ \phi^n(t)\ dt. \tag{1.14}$$

Several substitutions will be made in the integrands on the right hand side of (1.14). In the first and third integrals let $\sigma_\phi^{\,n}(t)$ be replaced by $\sigma_\phi^{\,n\,*}(t)$ and $\dot{\sigma}_\phi^{\,n}(t)$ by $\dot{\sigma}_\phi^{\,n\,*}(t)$ respectively where

$$\sigma_\phi^{\,n\,*}(t) = -\ F^{-1}\ [G(j\omega)\ F[\phi^n(t)]]$$

and

$$\dot{\sigma}_\phi^{\,n\,*}(t) = -F^{-1}\ [j\ \omega\ G(j\omega)\ F[\phi^n(t)]]$$

with F and F^{-1} denoting the direct and inverse Fourier transform operations, respectively. The values of these integrals are unchanged since the starred quantities are equal to their unstarred counterparts in $(0, T_n)$. The value of $\sigma_\phi^{\,n\,*}(t)$ for $t > T_n$ does not affect the first integral since $\delta(t) + y(t) = 0$ for $t < 0$ and $\phi^n(t) = 0$ for $t > T_n$. The latter reason also shows that the third integral is not influenced by the values of $\dot{\sigma}_\phi^{\,n\,*}(t)$ for $t > T_n$. In the case of the second integral

$x(t)$ being non-zero for $t < 0$ implies that $\sigma_\phi{}^n(t)$ cannot be replaced by $\sigma_\phi{}^{n\,*}(t)$ without changing the value of this integral. Therefore, the portion of $\sigma_\phi{}^{n\,*}(t)$ for $t > T_n$ must be taken into account in making the substitution. Let

$$\sigma_\phi{}^{n\,*}(t) = \sigma_\phi{}^n(t) + \sigma_\phi{}^d(t) \tag{1.15}$$

where $\sigma_\phi{}^d(t)$ is that component of $\sigma_\phi{}^{n\,*}(t)$ occurring in (T_n,∞). With these substitutions the first three integrals on the right hand side of (1.14) are

$$\int_o^{T_n} ((\delta(t) + y(t)) * (\sigma_\phi{}^{n*}(t) - \phi^n(t)/k_2))\ \phi^n(t)\ dt +$$

$$\int_o^{T_n} (x(t) * (\sigma_\phi{}^{n*}(t) - \phi^n(t)/k_2))\ \phi^n(t) dt + \alpha \int_o^{T_n} \dot{\sigma}_\phi{}^{n*}(t)\ \phi^n(t)\ dt$$

$$- \int_o^{T_n} (\sigma_\phi{}^d(t) * x(t))\ \phi^n(t)\ dt\ . \tag{1.16}$$

For the final step in the proof a bound is required on the last integral of (1.16) in terms of $|\phi^n(t)|max$, the largest value of $|\phi^n(t)|$ in $(0,T_n)$. $|\sigma_\phi{}^d(t)|$ is given by

$$\left| \int_o^t g(\lambda)\ \phi^n(t-\lambda)\ d\lambda \right|\ ,\ t \geq T_n\ , \tag{1.17}$$

where $g(\lambda) = F^{-1}(G(j\omega))$. Because of condition b of the theorem, it is possible to find two positive numbers q and r such that $|g(\lambda)| < q\ \exp(-r\lambda)$. Using this bound gives

$$|\sigma_\phi{}^d(t)| \leq \int_{t-T_n}^t q\ \exp(-r\lambda)\ |\phi^n(t)|_{max}\ d\lambda =$$

$$(q/r)\ |\phi^n(t)|_{max}\ \exp(-rt)\ [\exp(rT_n)-1],\ t \geq T_n\ . \tag{1.18}$$

The lower limit on the integral has been changed to $t - T_n$ since $\phi^n(t)$ is zero outside $(0,T_n)$.

The piecewise continuous and impulsive components of x(t) will be considered separately. Since $|x'(t)| < \ell \exp(ft)$, using (1.18) gives

$$|\sigma_\phi^d(t) * x'(t)| \le$$

$$\frac{\ell q}{r} |\phi^n(t)|_{max} \int_{-\infty}^{-T_n+t} \exp(f\lambda) \exp(-r(t-\lambda))[\exp(rT_n)-1]d\lambda$$

$$= \frac{\ell q}{r(r+f)} |\phi^n(t)|_{max} [\exp(rT_n)-1] \exp(-(r+f)T_n) \exp(ft) \quad (1.19)$$

$$0 \le t \le T_n.$$

Using this result gives

$$\int_0^{T_n} (\sigma_\phi^d(t) * x'(t))\, \phi^n(t)dt \le |\phi^n(t)|_{max} \int_0^{T_n} |\sigma_\phi^d(t) * x'(t)|dt \le$$

$$\frac{\ell q|\phi^n(t)|^2_{max}}{rf(r+f)} (1 - \exp(-rT_n))(1 - \exp(-fT_n)) \le M_1|\phi^n(t)|^2_{max} \quad (1.20)$$

where M_1 is a positive number independent of T_n.

For the impulsive case,

$$\sigma_\phi^d(t) * x(t) = \sum_{i=1}^{u} a_i\, \sigma_\phi^d(t+b_i) \quad (1.21)$$

and

$$\left| \int_0^{T_n} (\sigma_\phi^d(t) * x(t))\, \phi^n(t)dt \right| \le \sum_{i=1}^{u} |a_i| \int_0^{T_n} |\sigma_\phi^d(t+b_i)\phi^n(t)|dt. \quad (1.22)$$

If $b_i < T_n$, the use of (1.18) in the right hand side integral of (1.22) gives

$$\int_0^{T_n} |\sigma_\phi{}^d(t+b_i)\phi^n(t)dt| \le \frac{q}{r}\,|\phi^n(t)|^2_{max}\,[\exp(rT_n)-1]\int_{T_n-b_i}^{T_n} \exp(-r(t+b_i))dt$$

(1.23)

The lower limit on the right hand side integral is $T_n - b_i$ since $\sigma_\phi{}^d(t+b_i) = 0$ for $t < b_i - T_n$. Evaluating (1.23) gives

$$\frac{q}{r^2}\,|\phi^n(t)|^2_{max}\,[1-\exp(-rT_n)][1-\exp(-rb_i)] \le M_{2i}|\phi^n(t)|^2_{max} \tag{1.24}$$

where M_{2i} is a positive number. Finally, if $b_i > T_n$, the left hand side of (1.23) is less than or equal to

$$\frac{q}{r}\,|\phi^n(t)|^2_{max}[\exp(rT_n)-1]\int_0^{T_n} \exp(-r(t+b_i))dt = \frac{q}{r^2}|\phi^n(t)|^2_{max} \; X$$

$$[\exp(-r(b_i - T_n)) - \exp(-r\,b_i)][1 - \exp(-r\,T_n)] \le M_{3i}|\phi^n(t)|^2_{max}$$

(1.25)

where M_{3i} is a positive number. Using (1.20), (1.24), and (1.25) gives

$$\left|\int_0^{T_n} (\sigma_\phi{}^d(t) * x(t))\,\phi^n(t)dt\right| \le (M_1 + \sum_{i=1}^{u} |a_i|M_3)\;|\phi^n(t)|^2_{max}$$

$$= M|\phi^n(t)|^2_{max} \tag{1.26}$$

where M_3 is the largest of the M_{2i}'s and M_{3i}'s and M is a positive number independent of T_n. That is the desired bound.

Since $\phi^n(t)$ is zero outside $(0,T_n)$, the limits on the first 3 integrals of (1.16) may be changed to $(-\infty,\infty)$. Also, because of the conditions on the various functions involved, Parseval's Theorem is applicable to these integrals. Its application gives

$$\int_{-\infty}^{+\infty} ((\delta(t) + x(t) + y(t)) * (\sigma_{\phi}^{n*}(t) - \phi^{n}(t)/k_2))\ \phi^{n}(t)\ dt +$$

$$\alpha \int_{-\infty}^{+\infty} \dot{\sigma}_{\phi}^{n*}(t)\ \phi^{n}(t)dt = -\ \frac{1}{2\pi} \int_{-\infty}^{+\infty} ((1+X(j\omega) + Y(j\omega))(G(j\omega) + 1/k_2)$$

$$+ \alpha j\ \omega\ G(j\omega))\ \ |F[\phi^{n}(t)]|^2 d\omega \qquad (1.27)$$

Since the imaginary part of the integral on the right hand side of (1.27) is zero, (1.27) may be rewritten as

$$-\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{Re}(1 + X(j\omega) + Y(j\omega) + \alpha j\omega)(G(j\omega) + 1/k_2)\ |F[\phi^{n}(t)]|^2 d\omega. \qquad (1.28)$$

From (1.1) it follows that (1.28) is non-positive. Combining (1.4), (1.14), (1.16), (1.26), (1.27), and (1.28) gives

$$c(T_n) \int_{o}^{T_n} (\sigma^{n}(t) - \phi^{n}(t)/k_2)\dot{\phi}^{n}(t)dt + \alpha\Phi(T_n) - \alpha\Phi(0) \leq$$

$$\left|\int_{o}^{T_n} (\sigma_{\phi}^{d}(t) * x(t))\ \phi^{n}(t)dt\right| + \left|\int_{o}^{T_n} (\sigma_{i}^{n}(t) + (x(t) * \sigma_{i}^{n}(t)) +\right.$$

$$\left.(y(t) * \sigma_{i}^{n}(t)) + \alpha\dot{\sigma}_{i}^{n}(t))\ \phi^{n}(t)\ dt\right| \qquad (1.29)$$

$$\leq M|\phi^{n}(t)|^2_{max} + P|\phi^{n}(t)|_{max} \qquad (1.30)$$

where

$$P = \int_{o}^{\infty} |\sigma_{i}(t) + x(t) * \sigma_{i}(t) + y(t) * \sigma_{i}(t) + \alpha\dot{\sigma}_{i}(t)|dt$$

and

$$\Phi(T_n) = \int_o^{\sigma(T_n)} \phi(\sigma)\, d\sigma.$$

Therefore,

$$\Phi(T_n) \leq \frac{1}{\alpha} [M\, |\phi^n(t)|^2_{max} + P|\phi^n(t)|_{max}] + \Phi(0). \tag{1.31}$$

Using the approach given in Lefschetz [11], let T_n be chosen such that $|\phi^n(t)|_{max}$ occurs at T_n. Then with the first part of condition c holding, it follows that σ and hence $\phi(\sigma)$ are bounded; if this were not the case, inequality (1.31) would not hold for large values of $|\sigma|$. If the second part of condition c holds, a quadratic Liapunov function may be found using the approach of Rekasius [12] that shows the boundedness of σ and $\phi(\sigma)$.

Since the right hand side of (1.31) is bounded, it follows from (1.30) that $\int_o^{T_n} (\sigma^n(t) - \phi^n(t)/k_2)\, \phi^n(t)dt$ is bounded, from which asymptotic stability in the large follows, using the arguments given in Aizerman and Gantmacher [13]. This completes the proof of the theorem.

In order to prove corollary 1, the lemma is applied directly to (1.8) to give $|I(T)| \leq I(0)$ instead of $I(T) \leq I(0)$. (1.13) then becomes

$$\left| \int_o^{T_n} ((x(t) + y(t)) * (\sigma^n(t) - \phi^n(t)/k_2))\, \phi^n(t)dt \right| \leq$$

$$\left[\int_{-\infty}^{+\infty} (|x'(\lambda)| + |y'(\lambda)|)d\lambda + \sum_{i=1}^{u} |a_i| + \sum_{j=1}^{v} |c_i|\right] I(0). \tag{1.32}$$

Using the condition of this corollary, it follows that the left hand side of (1.32) is less than or equal to I(0), from which (1.4) follows. The remainder of the proof is unchanged. This completes the proof of corollary 1.

To prove the assertion of corollary 2, it is first shown that if

$$\text{Re } Z(G + 1/k_2) \geq \delta_2 > 0,$$

$$\text{Re } Z(G/(1 + \varepsilon G) + 1/k_2) \geq \delta_3 > 0$$

for ε sufficiently small. δ_3 is a positive number. By a straightforward calculation Re $Z(G/(1 + \varepsilon G) + 1/k_2)$ is

$$\frac{\text{Re } Z\ (G + 1/k_2) + \varepsilon(\text{Re } Z\ )\ [|G|^2\ (1 + \varepsilon/k_2) + 2(\text{Re } G)/k_2]}{(1 + \varepsilon R)^2 + (\varepsilon X)^2}$$

The first quantity in the numerator is non-negative. Since Re Z is non-negative, the second quantity in the numerator may be negative if $-2/(k_2 + \varepsilon) < \text{Re } G < 0$. For this interval ε must be chosen small enough such that the numerator is positive. This is guaranteed by having

$$\varepsilon < \frac{-\delta_2 k_2}{2\text{Re } Z \quad \text{Re } G}$$

in the interval. Let the linear transformation $\phi_1(\sigma) = \phi(\sigma) - \varepsilon\sigma$ be applied to the system. Then $G_1 = G/(1 + \varepsilon\ G)$. The stability of the transformed system will guarantee the stability of the original system. If ε is chosen to be less than both δ and the right hand side of the ε inequality, the transformed system will satisfy the conditions of the theorem for the noncritical cases. Q.E.D.

The proof of corollary 3 follows directly from the proof of the theorem with $x(t) = 0$. (1.31) becomes $\Phi(T_n) \leq \frac{P}{\alpha} |\phi^n(t)|_{max} + \Phi(0)$. Since $\phi(\sigma)$ is a monotone increasing function of σ, for $|\sigma|$ sufficiently large the left hand side of this inequality will become greater than the right, showing that $\sigma(t)$ and $\phi(\sigma(t))$ are bounded. The remainder of the proof is unchanged.

F. Theorem for a Nonlinearity With a Monotone Bound

This theorem is an improved version of one given in [4]. The two improvements consist of permitting $Z(s)$ to have a corresponding time function that is non-zero for $t < 0$ and of taking the symmetry of the nonlinearity into account, resulting in $x(t)$ and $y(t)$ being allowed to take on positive as well as negative values.

Theorem 1.2. For the system given in figure 1 let the following conditions hold:

a. $A\phi_m(\sigma)\,\sigma \leq \phi(\sigma)\,\sigma \leq B\phi_m(\sigma)\,\sigma$, where A and B are real numbers satisfying $0 < A \leq 1$ and $1 \leq B < \infty$, $\phi(0) = \phi_m(0) = 0$, $\sigma\,\phi(\sigma) < k\,\sigma^2$ where $k > 0$ and $\sigma\,\phi_m(\sigma) > 0$ for $\sigma \neq 0$, $d\phi(\sigma)/d\sigma$ is a continuous function of σ, $\phi_m(\sigma)$ is a continuous monotone increasing function of σ having an odd part $\phi_{mo}(\sigma)$ that satisfies $|\phi_m(\sigma)| \leq C|\phi_{mo}(\sigma)|$ and $|\phi_{mo}(\sigma)| \leq D|\phi_m(\sigma)|$.

b. Conditions b and c of theorem 1.1.

Then a sufficient condition for asymptotic stability in the large is that

$$\text{Re}\ [Z(j\omega)\ G(j\omega) + E\ (G(j\omega) + 1/k)] \geq 0 \tag{1.33}$$

for all real ω where E is a non-negative number. $Z(j\omega)$ is defined as in (1.2) but (1.3) becomes

$$\frac{BCD}{A}\ [\int_{-\infty}^{+\infty} (x'^{+}(t) + y'^{+}(t))\ dt + \sum a_i^{+} + \sum c_i^{+}] -$$

$$\frac{B}{A}\ [\int_{-\infty}^{+\infty} (x'^{-}(t) + y'^{-}(t))dt + \sum a_i^{-} + \sum c_i^{-}] < 1 \tag{1.34}$$

where $x'^{+}(t)$, y'^{+}, a_i^{+}, and c_i^{+} are the positive portions or values of the corresponding non-superscripted functions or numbers and $x'^{-}(t)$, $y'^{-}(t)$, a_i^{-}, and c_i^{-} are the negative portions or values of the corresponding non-superscripted functions or numbers.

Proof. Starting with (1.4) of the proof of theorem 1.1, let this equation be replaced by

$$\int_{o}^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt = c(T_n)\int_{o}^{T_n} \sigma^n(t)\phi^n(t)dt \tag{1.35}$$

as the condition to be shown. Repeating the steps used to obtain (1.6) and (1.7) gives

$$\int_{-\infty}^{o} (x'^{+}(\lambda) + x'^{-}(\lambda)) \int_{o}^{T_n} \sigma(t - \lambda)\ \phi^n(t)dt\ d\lambda +$$
$$\int_{o}^{\infty} (y'^{+}(\lambda) + y'^{-}(\lambda)) \int_{o}^{T_n} \sigma(t - \lambda)\ \phi^n(t)\ dt\ d\lambda \tag{1.36}$$

and

$$\sum a_i^+ \int_0^{T_n} \sigma^n(t + b_i)\, \phi^n(t)dt + \sum a_i^- \int_0^{T_n} \sigma^n(t + b_i)\phi^n(t)\, dt$$

$$+ \sum c_i^+ \int_0^{T_n} \sigma^n(t-d_i)\, \phi^n(t)dt + \sum c_i^- \int_0^{T_n} \sigma^n(t-d_i)\phi^n(t)dt. \quad (1.37)$$

$I(T)$ then becomes $I(T) = \int_0^{T_n} \sigma^n(t - T)\, \phi^n(t)dt$. At this point the proof differs from that of theorem 1 for it is desired to develop both positive and negative bounds on $I(T)$. First a bound is developed on $|I(T)|$.

$$|I(T)| \leq B \int_0^{T_n} |\sigma^n(t - T)\, \phi_m{}^n(t)|dt$$

$$\leq BC \int_0^{T_n} |\sigma^n(t - T)\, \phi_{mo}{}^n(t)|dt \leq BC \int_0^{T_n} \sigma^n(t)\, \phi_{mo}{}^n(t)dt$$

$$(1.38)$$

where use has been made of the lemma. $\phi_m{}^n(t) = \phi_m{}^n(\sigma(t))$ and $\phi_{mo}{}^n(t) = \phi_{mo}{}^n(\sigma(t))$. Continuing the development gives

$$BC \int_0^{T_n} \sigma^n(t)\, \phi_{mo}{}^n(t)dt \leq BCD \int_0^{T_n} \sigma^n(t)\, \phi_m{}^n(t)$$

$$\leq \frac{BCD}{A}\int_0^{T_n} \sigma^n(t)\, \phi^n(t)dt\ . \quad (1.39)$$

The negative bound on $I(T)$ is then

$$I(T) \geq -\frac{BCD}{A} I(0)\ . \quad (1.40)$$

For the positive bound the same procedure as in theorem 1 is used to give

$$\int_0^{T_n} \sigma^n(t-T)\phi^n(t)dt \le \int_0^{T_n} \sigma_+^n(t-T)\ \phi_+^n(t)dt + \int_0^{T_n} \sigma_-^n(t-T)\phi_-^n(t)dt$$

$$\le B \int_0^{T_n} \sigma_+^n(t-T)\ \phi_{m+}{}^n(t)dt + B\int_0^{T_n} \sigma_-^n(t-T)\ \phi_{m-}{}^n(t)dt$$

$$\le B\int_0^{T_n} \sigma^n(t)\ \phi_m{}^n(t)dt \le \frac{B}{A} \int_0^{T_n} \sigma^n(t)\ \phi^n(t)dt\ . \qquad (1.41)$$

Using these two bounds in (1.36) and (1.37) gives

$$\int_0^{T_n} ((x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt \ge$$

$$-\frac{BCD}{A}\ [\int_{-\infty}^{+\infty} (x'^+(\lambda) + y'^+(\lambda))\ d\lambda + \sum a_i{}^+ + \sum c_i{}^+]\ I(0)$$

$$+\frac{B}{A}\ [\int_{-\infty}^{+\infty} (x'^-(\lambda)+y'^-(\lambda))d\lambda + \sum a_i{}^- + \sum c_i{}^-]\ I(0)\ . \qquad (1.42)$$

Using (1.42) and (1.34) gives (1.35). The remainder of the proof is similar to that of theorem 1 with the left hand side of (1.14) replaced by

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt + \alpha\int_0^{T_n} \dot\sigma^n(t)\ \phi^n(t)dt$$

$$+ E\int_0^{T_n} (\sigma^n(t) - \phi^n(t)/k)\ \phi^n(t)dt\ . \qquad (1.43)$$

Q.E.D.

The frequency domain condition (1.33) is certainly not as easy to apply as (1.1), (1.33) was obtained because of the necessity of using (1.35) in order to apply the various conditions on $\phi(\sigma)$. An example of the application of this theorem is considered next.

Example 3. Let $\phi(\sigma)$ be an odd function defined for positive values of σ by

$$\begin{aligned}\phi(\sigma) &= \sigma &&, 0 \leq \sigma \leq 1.25\\ &= -\sigma + 2.5 &&, 1.25 \leq \sigma \leq 1.5\\ &= (5\sigma/3)/(1+\sigma) &&, 1.50 \leq \sigma\end{aligned}$$

and let $G(s) = K(s + 4)(s + 50)^2/(s + .1)(s + 1)(s + 1000)^2$, with K being large but finite. It is assumed that the kinks in the $\phi(\sigma)$ curve are smoothed out so that the derivative is a continuous function of σ. A plot of this nonlinear characteristic reveals that a convenient choice is to take $\phi_m(\sigma)$ as an odd function equal to $\phi(\sigma)$ for positive values of σ except for $1.25 \leq \sigma \leq 3.01$ for which interval $\phi_m(\sigma) = 1.25$. $\phi_m(\sigma)$ is then a continuous odd monotone increasing function of σ. With this choice A = .8, B = C = D = 1 and (1.34) becomes $\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)dt < .8$. Since K is to be large but finite, let E = 0 to give Re $Z(j\omega)\, G(j\omega) \geq 0$ as the criterion to be satisfied. $G(j\omega)$ has an angle that lies outside the $\pm\, 90^\circ$ band in a lagging direction at low frequencies, at higher frequencies the angle approaches $+\, 90^\circ$ and then $-\, 90^\circ$ at very high frequencies. Because of this behavior, the Popov criterion will not show stability. Let $Z(s) = (s + 1)(s + 1000)/(s + 4)$. This particular function has the proper phase characteristic, that is, leading at low frequencies, almost zero at intermediate frequencies, and then leading at high frequencies to give a product with an angle in the $\pm 90^o$ band. Since $Z(s)\ G(s) = K\ (s + 50)^2/(s + .1)(s + 1000)$, it is seen that $\mathrm{Re}\ Z(j\omega)\ G(j\omega) > 0$ for all ω. Expressing $Z(s)$ in a partial fraction expansion form gives $Z(s) = s + 997 - 2988/(s + 4)$. The left hand side of (J.34) is .937, and hence this condition is satisfied. Therefore, the given system is asymptotically stable in the large.

G. Conclusion

This chapter has presented two theorems which allow the $Z(s)$ multiplier to correspond to a function of time that is non-zero for $t < 0$ as well as for $t > 0$. This innovation solves the problem of obtaining a $Z(j\omega)$ whose angle varies with equal freedom between $0^o - +90^o$ and $0^o - -90^o$. The generalized RL $Z(s)$ multiplier considered shows that a nonlinear system having a monotone non-linearity with a slope in the sector $(0,k_2)$ is stable provided that the system is stable for linear gains in the sector $(0,k_2)$ and provided that the angle changes slowly enough with frequency. Although this work gives improved results, it is not clear how close these results are to the actual absolute stability limit. Additional study is needed to resolve this matter.

While the two Z(s) functions discussed appear to be quite useful, if it is not possible to show stability with either of these two, it is not clear how one should go about generating additional Z(s) functions with more desirable characteristics, other than to use trial and error. The reason for this is the need to consider simultaneously both the time and the frequency domain behavior of a possible candidate for a Z(s) function. This appears to be a worthwhile area for further research.

Condition c of theorem 1.1 is one way of guaranteeing the boundedness of $\sigma(t)$ and $\phi(t)$. If a certain nonlinearity does not satisfy this condition, the theorem may still be applied provided that a Liapunov function can be found that will show the boundedness of the state variables of the system. However, finding a suitable Liapunov function may be a difficult task.

II. Appendix 1

Lemma. If $f_a(t)$ and $f_b(t)$ are two continuous time functions which are zero outside the time interval $(0, T_n)$, $f_b(t) = h(f_a(t))$ where h is a piecewise continuous monotone increasing function of f_a, and if either $f_a(t)$ and $f_b(t)$ are both always non-negative or non-positive or h is an odd monotone function with $h(0) = 0$, then

$$\int_o^{T_n} (f_a(t)\ f_b(t) - |f_a(t)\ f_b(t + T)|)dt \geq 0$$

for any real value of T.

Proof. Given a value of $T > 0$, let the summation

$$\sum_{i=1}^{n} |f_a(\delta i)\ f_b(\delta i + T)|\ \delta \tag{A1}$$

be formed where δ is a positive number chosen such that T/δ is an integer and n is chosen such that $n\delta = T_n - \delta_1$ where δ_1 is a positive number less than δ. Let a ranking of the magnitudes of the values of $f_a(t)$ and $f_b(t)$ that can appear in the summation be set up such that $|f_{a1}| \geq |f_{a2}| \geq |f_{a3}|\ldots$ for f_a and a similar ordering $|f_{b1}| \geq |f_{b2}| \geq |f_{b3}|\ldots$ holds for f_b. Since h is monotone increasing and either an odd function or $f_a(t)$ and $f_b(t)$ are both always non-positive or non-negative, values of $|f_{ai}|$ and $|f_{bj}|$ with the same numerical subscript occur at the same time or the ranking can be arranged such that they occur at the same time if two or more magnitudes are equal. Using the ranked magnitudes, a table of product values that may appear in the summation is formed as indicated below.

$$\begin{array}{lllllll}
 & |f_{b1}| & |f_{b2}| & |f_{b3}| \cdots & |f_{bj}| \cdots & |f_{bn}| \\
|f_{a1}| & f_{a1}f_{b1} & |f_{a1}f_{b2}| & & & \\
|f_{a2}| & |f_{a2}f_{b1}| & f_{a2}f_{b2} & & & \\
|f_{a3}| & & & & & \\
\vdots & & & & & \\
|f_{ai}| & & & & & \\
\vdots & & & & & \\
|f_{an}| & & & & & f_{an}f_{bn}
\end{array}$$

The diagonal elements in this table correspond to the terms that appear in (A1) with T = 0. For any value of T, the terms $|f_{ai}|$ and $|f_{bj}|$ can appear only once, if at all, in the summation. This means that of the product elements appearing in (A1), only one element can occur in a given row and one element in a given column in the table of product values. Also, for $T \neq 0$, the summation terms appear as off diagonal elements in the table. Next, by using a row and column counting process it will be shown that

$$\sum_{i=1}^{n} f_a(\delta i)\ f_b(\delta i) \geq \sum_{i=1}^{n} |f_a(\delta i)\ f_b(\delta i + T)| \qquad (A2)$$

for $T \neq 0$.

Consider the elements on the right hand side of (A2) that appear in the first row or first column of the table of product values. The maximum possible number is two. If it is zero or one, an inequality $f_{a1}f_{b1} \geq 0$, $f_{a1}f_{b1} \geq |f_{a1}f_{bj}|$ or $f_{a1}f_{b1} \geq |f_{ai}f_{b1}|$ is formed. The first row and the first column are then removed, giving a reduced table of product values. If there are two elements, it it necessary to consider three cases.

a. The two terms are $|f_{aj}f_{b1}|$ and $|f_{a1}f_{bj}|$. In this case the two diagonal terms $f_{a1}f_{b1}$ and $f_{aj}f_{bj}$ are used to give the inequality $f_{a1}f_{b1} + f_{aj}f_{bj} \geq |f_{aj}f_{b1}| + |f_{a1}f_{bj}|$. Since the only two elements possible in the first and jth rows and columns have been bounded by the diagonal terms associated with these rows and columns, the first and jth rows and columns are removed, giving a reduced table of product values.

b. The two terms are $|f_{ai}f_{b1}|$ and $|f_{a1}f_{bj}|$ with $i < j$. An inequality that may be written is $f_{a1}f_{b1} + f_{ai}f_{bi} \geq |f_{ai}f_{b1}| + |f_{a1}f_{bi}|$. If there is no term in the ith column, $|f_{a1}f_{bi}|$ is used to bound $|f_{a1}f_{bj}|$, since $|f_{a1}f_{bi}| \geq |f_{a1}f_{bj}|$, giving as the desired inequality $f_{a1}f_{b1} + f_{ai}f_{bi} \geq |f_{ai}f_{b1}| + |f_{a1}f_{bj}|$. The first and ith rows and columns are then removed to give a reduced table of product values. If there is a term in the ith column, say $|f_{ak}f_{bi}|$, the $|f_{ak}f_{bi}|$ and $|f_{a1}f_{bj}|$ terms are bounded by the $|f_{a1}f_{bi}|$ term and the $|f_{ak}f_{bj}|$

term, giving the inequality $|f_{a1}f_{bi}| + |f_{ak}f_{bj}| \geq |f_{a1}f_{bj}| + |f_{ak}f_{bi}|$. Combining this bound with the one involving $|f_{ai}f_{b1}|$ gives $f_{a1}f_{b1} + f_{ai}f_{bi} + |f_{ak}f_{bj}| \geq |f_{ai}f_{b1}| + |f_{a1}f_{bj}| + |f_{ak}f_{bi}|$ as the overall inequality resulting from this step. The $|f_{ak}f_{bj}|$ term has been borrowed to obtain the bound. This term is not an element of the summation since the kth row and jth columns by hypothesis each have one element. A reduced table of product values is obtained by deleting the first and ith rows and columns and adding the $|f_{ak}f_{bj}|$ term as one to be bounded by the remaining diagonal elements. The array obtained has the same properties as the original array with regard to each row and column having only one element. Therefore, the process may be repeated on the reduced product value table.

c. The two terms are $|f_{ai}f_{b1}|$ and $|f_{a1}f_{bj}|$ with $i > j$. The strategy of b is repeated with the roles of the ith and jth column being taken by the jth and ith rows, respectively. The process is then applied to the first row and column of the reduced table of product values and repeated until there are no terms left in the final reduced table. Adding together the inequalities obtained at each stage of the process gives the left hand side of (A2) plus additional terms greater than the right hand side of (A2) plus the same additional terms. Upon cancelling the common terms, (A2) results. From (A2) it follows that

$$\delta \sum_{i=1}^{n} (f_a(\delta i)\, f_b(\delta i) - |f_a(\delta i)\, f_b(\delta i + T)|) \geq 0.$$

Since

$$\int_o^{T_n} (f_a(t)f_b(t) - |f_a(t)f_b(t + T)|)dt =$$

$$\sum_{i=1}^{n} (f_a(\delta i)\ f_b(\delta i) - |f_a(\delta i)f_b(\delta i + T)|)\delta + F,$$

where F is a real number that can be made arbitrarily small by a suitable choice of δ, taking the limit as $\delta \to o$ gives the assertion of the lemma for positive T. A similar discussion shows that the lemma also holds for negative T. Q.E.D.

IV CHAPTER II. BOUNDS ON THE RESPONSE OF AN AUTONOMOUS SYSTEM WITH A SINGLE NONLINEARITY

A. Introduction

This chapter is concerned with the calculation of bounds on the response of the single nonlinearity system of Figure 1. For the first theorems it is assumed that the external input to the system is zero and that the system is excited by initial conditions only. Then, Fourier transformable inputs of a certain class are permitted in later theorems. If the input is itself bounded, the bounds which are calculated on the response enable the showing of Liapunov stability but not asymptotic stability. The bound that is determined is on the function $\Phi(\sigma(t))$ and usually takes one of the forms shown in figure 6. Once a bound has been obtained on $\Phi(\sigma(t))$, a bound can be calculated for $\sigma(t)$ for specific nonlinear characteristics.

Pertinent references include the survey paper by Kalman and Bertram [14] in which it is pointed out that an exponential bound can be obtained on the response by the use of Liapunov functions. The maximum value of $\dot{v}/v = -\eta$ is calculated over the space in which the response is confined. The bound is then $v(t) \leq v(0)\, e^{-\eta t}$. The bound on $v(t)$ can then be converted into a bound on the system variables. Sandberg [15] considered the problem of a time varying nonlinearity confined to a linear sector and gave a frequency domain condition

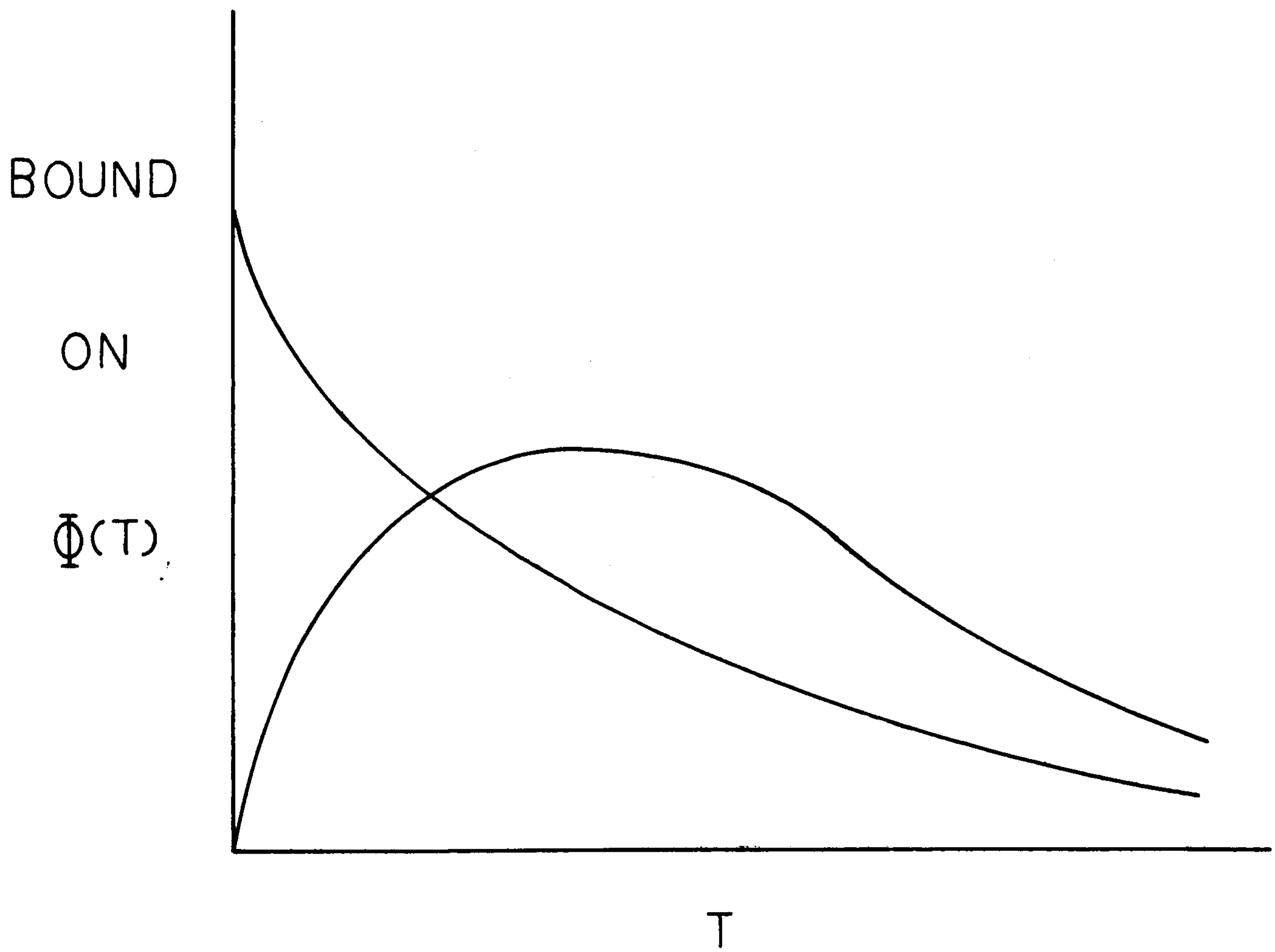

Figure 6. Typical Bounds on $\Phi(\sigma(t))$.

guaranteeing that the state variables approach zero exponentially with time. In the single stationary nonlinearity case with a zero lower bound on the nonlinearity, this frequency domain condition is equivalent to Re $G(j\omega) \geq 0$, which is a rather restricted criterion. Tsypkin [16] obtained an analogous result using a Popov type approach for a sampled data system having a single nonlinearity. Using a Liapunov approach, Yakubovich [17] showed that for a nonlinearity confined to a sector $(0, k)$, if Re $G(j\omega - a)(1 + \alpha j\omega) + 1/k \geq 0$, then the response of the system satisfies $|\sigma(t)| \leq Me^{-at}|\sigma(0)|$ where M is a positive number. This last result is similar to the Popov criterion except for the shift in the argument of $G(j\omega)$.

Although the criteria of the last 3 references show the existence of a bound of the desired type, these references do not consider the problem of calculating a value of M. Also, the corresponding frequency domain stability criteria for these works are more restricted than those given in chapter 1. Therefore, the main object of this paper is to develop theorems giving bounds on the response of systems using the approach employed in the development of the stability criteria of chapter 1. Once a system has been shown to be asymptotically stable in the large using these criteria, it will then be possible to calculate a bound on the response using the results of this chapter.

The first three theorems deal with those systems in which it is possible to show stability with $x(t) = 0$. Theorems 2.4 and 2.5 give bounding expressions for those cases in which $x(t) \neq 0$. Since for this case a bound must be available on the response of the system of the form $\Phi(t) \leq M|\phi(t)|_{max}$, where M is a positive number and $|\phi(t)|_{max}$ is the largest value of $\phi(t)$ in $(0, T_n)$, the application of these latter two theorems requires somewhat more computation than the first 3. The bounds for these first five theorems are calculated using a "completing the square" approach of Aizerman and Gantmacher [13]. Under certain circumstances an improved bound can be found using the approach of Lefschetz [11]. This is used in theorem 2.6 and 2.7. Theorem 2.8 gives a bound on the response with an external input applied and theorem 2.9 considers a special case which arises when dealing with systems having lag compensators. Finally, the possibility of obtaining an improved bound when the system is in the linear region is discussed.

B. The Theorems

Theorem 2.1. For the system of figure 1 excited by initial conditions only let the following hold:

a. $0 \leq d\phi(\sigma)/d\sigma \leq k_2$ where k_2 is a positive number, $\phi(\sigma)$ and $(\sigma - \phi(\sigma)/k_2) = 0$ only for $\sigma = \phi(\sigma) = 0$, and $d\phi(\sigma)/d\sigma$ be a continuous function of σ.

b. $G(s) = N(s)/D(s)$ with the degree of $N(s)$ at least one less than the degree of $D(s)$ and with the zeros of $D(s)$ having negative real parts whose magnitude is greater than or equal to the positive number a.

c. $$\text{Re } H(j\omega) = \text{Re}[c(1 + Y(j\omega))(G(j\omega - a) + 1/k_2) + dj\omega\, G(j\omega - a) + adG(j\omega - a)] \geq b > 0$$

where b, c, and d are positive numbers, $y(t)$ is composed of delayed impulses and a piecewise continuous function that satisfies $y(t) \leq 0$ for $t > 0$, $y(t) = 0$ for $t < 0$ and

$$\int_0^\infty |y(t)|\, e^{at} dt < 1 . \tag{2.1}$$

Then

$$\Phi(T_n) = \int_0^{\sigma(T_n)} \phi(\sigma) d\sigma \leq e^{-2aT_n} \left[\frac{\int_0^\infty m^2(t)dt}{4d} + \int_0^{\sigma(0)} \phi(\sigma) d\sigma\right] \tag{2.2}$$

where $m(t) = F^{-1}\, [P(j\omega)\, Q(j\omega)]$ with

$$p(t) = e^{at}[(c + 2ad)\, \sigma_i^{\,n}(t) + d\, \dot{\sigma}_i^{\,n}(t)] + c(\sigma_i^{\,n}(t)\, e^{at} * y(t))^n$$

and $Q(j\omega)$ is defined by $1/\text{Re } H(j\omega) = Q(j\omega)\, Q(-j\omega)$. $\sigma_i^{\,n}(t)$ is equal to the initial condition component of $\sigma(t)$, $\sigma_i(t)$, in $(0, T_n)$ and zero outside this interval. Similarly, $\dot{\sigma}_i^{\,n}(t)$ is equal to the initial condition component of $\dot{\sigma}_i(t)$ in $(0, T_n)$ and zero elsewhere. $(\sigma_i^n(t) * y(t))^n$ is equal to $\sigma_i^n(t)$ convolved with $y(t)$ in $(0, T_n)$ and zero elsewhere.

<u>Proof</u>. First it is desired to establish the non-negativeness of certain integrals which play a prominent role in the development. Using integration by parts with $\dot{\sigma}(t)\ \phi(t)$ being integrated gives

$$\int_0^{T_n} e^{2at}\ \dot{\sigma}(t)\ \phi(t)dt = e^{2aT_n}\ \Phi(T_n) - \Phi(0)$$

$$- 2a \int_0^{T_n} e^{2at}\ \Phi(t)dt\ . \tag{2.3}$$

Also,

$$2a \int_0^{T_n} e^{2at}\ \sigma(t)\phi(t)dt - 2a \int_0^{T_n} e^{2at}\ \Phi(t)dt \geq 0 \tag{2.4}$$

since $\sigma(t)\phi(t)$ and $\Phi(t)$ are both non-negative and $\Phi(t) = \int_0^{\sigma(t)} \phi(\sigma)d\sigma \leq \sigma(t)\phi(t)$ because of the monotone increasing property of $\phi(\sigma)$. Adding the first integral of (2.4) to both sides of (2.3) and rearranging gives

$$e^{2aT_n}\ \Phi(T_n) + 2a \int_0^{T_n} e^{2at}\ \sigma(t)\phi(t)dt - 2a \int_0^{T_n} e^{2at}\Phi(t)dt$$

$$= \int_0^{T_n} e^{2at}\ \dot{\sigma}(t)\phi(t)dt + 2a \int_0^{T_n} e^{2at}\ \sigma(t)\phi(t)dt + \Phi(0) \tag{2.5}$$

where the sum of the second and third terms on the left hand side of (2.5) are non-negative by (2.4).

The second relationship to be established is

$$\int_0^{T_n} e^{2at} (\sigma^n(t) - \phi^n(t)/k_2)\, \phi^n(t)\, dt \; +$$

$$\int_0^{T_n} e^{at} (y(t) * ((\sigma^n(t) - \phi^n(t)/k_2) e^{at}))\, \phi^n(t) dt \geq 0 \,. \tag{2.6}$$

Let the impulsive component of $y(t)$ be given by $\sum_{j=1}^{v} c_j\, \delta(t - d_j)$ where the c_j's are negative numbers and the d_j's positive numbers. Substituting this component into the second integral on the left of (2.6) and inserting an e^{-ad_j} inside the integral and e^{+ad_j} outside gives

$$\sum_{j=1}^{v} c_j\, e^{ad_j} \int_0^{T_n} e^{2a(t-d_j)} (\sigma^n(t - d_j) - \phi^n(t-d_j)/k_2) \phi^n(t) dt. \tag{2.7}$$

With the piecewise continuous component of $y(t)$, $y'(t)$, substituted into the same integral, the result is

$$\int_0^{T_n} e^{at}\, \phi^n(t) \int_0^{\infty} y'(\lambda)(\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\; e^{a(t-\lambda)} d\lambda\; dt \,. \tag{2.8}$$

Interchanging the order of integration and inserting an $e^{-a\lambda}$ inside the integration with respect to t and $e^{+a\lambda}$ outside gives

$$\int_0^{\infty} y'(\lambda)\; e^{a\lambda} \int_0^{T_n} e^{2a(t-\lambda)} (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\; \phi^n(t) dt\; d\lambda. \tag{2.9}$$

Appearing in both (2.7) and (2.9) is an integral of the form $\int_0^{T_n} e^{2a(t-\lambda)}(\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\; \phi^n(t) dt$ where $\lambda > 0$. This integral may be rewritten as

$$\int_0^{T_n} e^{2a(t-\lambda)} \left(\sigma_+^n(t-\lambda) - \phi_+^n(t-\lambda)/k_2\right) \phi_+^n(t)dt \; +$$

$$\int_0^{T_n} e^{2a(t-\lambda)} \left(\sigma_-^n(t-\lambda) - \phi_-^n(t-\lambda)/k_2\right)\phi_-^n(t)dt \; +$$

$$\int_0^{T_n} e^{2a(t-\lambda)} \left(\sigma_-^n(t-\lambda) - \phi_-^n(t-\lambda)/k_2\right)\phi_+^n(t)dt \; +$$

$$\int_0^{T_n} e^{2a(t-\lambda)} \left(\sigma_+^n(t-\lambda) - \phi_+^n(t-\lambda)/k_2\right)\phi_-^n(t)dt \; . \tag{2.10}$$

The plus subscript indicates that the function possessing it is equal to the non-subscripted function when the non-subscripted function is positive and zero otherwise. An analogous definition applies to the use of the negative subscript. For example, $\phi_-^n(t) = \phi^n(t)$ for $\phi^n(t) < 0$ and $\phi_-^n(t) = 0$ for $\phi^n(t) \geq 0$. (2.10) is certainly less than or equal to the first two integrals of this equation. Applying lemma 2 given in the appendix of this chapter to these two integrals gives that (2.10) is less than or equal to

$$\int_0^{T_n} e^{2at}\left(\sigma_+^n(t) - \phi_+^n(t)/k_2\right) \phi_+^n(t)\, dt \; +$$

$$\int_0^{T_n} e^{2at}\left(\sigma_-^n(t) - \phi_-^n(t)/k_2\right) \phi_-^n(t)dt = \int_0^{T_n} e^{2at}\left(\sigma^n(t) - \phi^n(t)/k_2\right)\phi^n(t)dt. \tag{2.11}$$

Using the positive bound (2.11) in (2.7) and (2.8) gives as a lower bound for the sum of these integrals

$$\left(\sum_{j=1}^{v} c_j e^{ad_j} + \int_0^{\infty} e^{at} y'(t)dt \right) \int_0^{T_n} e^{2at}(\sigma^n(t)-\phi^n(t)/k_2)\phi^n(t)dt \,. \tag{2.12}$$

Using (2.1) and (2.12) in (2.6) shows that (2.6) holds.

At this point the necessary time domain relationships have been obtained. The next step is to make use of Parseval's Theorem in converting the time domain integrals into corresponding integrals in the frequency domain.

Let $\sigma_\phi(t)$ and $\dot{\sigma}_\phi(t)$ be those components of $\sigma(t)$ and $\dot{\sigma}(t)$, respectively, due to the feedback signal $-\phi(t)$. Then

$$\begin{aligned}
& d \int_0^{T_n} e^{2at} \dot{\sigma}^n(t)\phi^n(t)dt + 2da \int_0^{T_n} e^{2at} \sigma^n(t)\phi^n(t)dt \\
& + c \int_0^{T_n} e^{at}((\delta(t) + y(t)) * [(\sigma^n(t) - \phi^n(t)/k_2)e^{at}]) \phi^n(t)dt = \\
& d \int_0^{T_n} e^{2at} \dot{\sigma}_\phi{}^n(t)\phi^n(t)dt + 2da \int_0^{T_n} e^{2at} \sigma_\phi{}^n(t)\phi^n(t)dt \\
& + c \int_0^{T_n} e^{at} ((\delta(t) + y(t)) * [(\sigma_\phi{}^n(t) - \phi^n(t)/k_2)e^{at}] \phi^n(t)dt \\
& + d \int_0^{T_n} e^{2at} \dot{\sigma}_i{}^n(t)\phi^n(t)dt + 2da \int_0^{T_n} e^{2at} \sigma_i{}^n(t)\phi^n(t)dt \\
& + c \int_0^{T_n} e^{at} ((\delta(t) + y(t)) * [\sigma_i{}^n(t)\, e^{at}]) \phi^n(t)dt \,.
\end{aligned} \tag{2.13}$$

In the first three integrals on the right hand side of (2.13) let $\sigma_\phi{}^n(t)$ be replaced by $\sigma_\phi{}^{n*}(t)$ and $\dot{\sigma}_\phi{}^n(t)$ by $\dot{\sigma}_\phi{}^{n*}(t)$ where

$$\sigma_\phi{}^{n*}(t) = F^{-1}[-G(j\omega)\ F(\sigma_\phi{}^n(t))]$$

and

$$\dot{\sigma}_\phi{}^{n*}(t) = F^{-1}\ [-j\omega G(j\omega)\ F(\sigma_\phi{}^n(t))]\ .$$

In the first two integrals since the starred and unstarred quantities are equal in $(0, T_n)$ and since $\phi^n(t)$ is zero outside $(0, T_n)$, this change can be made without altering the values of these integrals. For the third integral the identical reasoning plus $\delta(t) + y(t)$ being zero for $t < 0$ shows that the substitution can be made in this case also without changing the value of the integral. A second desired modification is to replace the 0, T_n limits on all 6 of the integrals on the right hand side of (2.13) by $-\infty$, ∞; once again this is justified by the nature of $\phi^n(t)$. This reasoning also allows the last substitution which is to be made in the third integral, namely the replacement of $((\delta(t) + y(t)) * [\sigma_i{}^n(t)e^{at}])$ by $((\sigma(t) + y(t)) * [\sigma_i{}^n(t)\ e^{at}])^n$. The second function is equal to the first in $(0, T_n)$ and zero elsewhere. With these changes (2.13) becomes

$$d \int_{-\infty}^{+\infty} e^{2at} \dot{\sigma}_{\phi}^{n*}(t)\phi^{n}(t)dt + 2da \int_{-\infty}^{+\infty} e^{2at} \sigma_{\phi}^{n*}(t)\phi^{n}(t)dt$$

$$+ c \int_{-\infty}^{+\infty} e^{at}((\delta(t) + y(t)) * [(\sigma_{\phi}^{n*}(t) - \phi^{n}(t)/k_{2})e^{at}])\phi^{n}(t)dt$$

$$+ d \int_{-\infty}^{+\infty} e^{2at} \dot{\sigma}_{i}^{n}(t)\phi^{n}(t)dt + 2da \int_{-\infty}^{+\infty} e^{2at} \sigma_{i}^{n}(t)\phi^{n}(t)dt$$

$$+ c \int_{-\infty}^{+\infty} e^{at} ((\delta(t) + y(t)) * [\sigma_{i}^{n}(t)e^{at}])^{n} \phi^{n}(t)dt . \qquad (2.14)$$

Applying the Parseval Theorem to (2.14) and using the fact that only the real parts of the first three integrands give a non-zero contribution to the values of these integrals gives

$$- \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Re } [d(j\omega - a) G(j\omega - a) + 2da\, G(j\omega - a)$$

$$+ c [1 + Y(j\omega)](G(j\omega - a) + 1/k_{2})] \, |F(\phi^{n}(t)e^{at})|^{2} \, d\omega$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} F[de^{at} \dot{\sigma}_{i}^{n}(t) + 2da\, e^{at} \sigma_{i}^{n}(t) +$$

$$+ c((\delta(t) + y(t)) * [\sigma_{i}^{n}(t)e^{at}])^{n}] \, \bar{F}(\phi^{n}(t)e^{at}) \, d\omega . \qquad (2.15)$$

Using c, the first integral can be rewritten as

$$- \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Re } H(j\omega) \, |F(\phi^{n}(t)e^{at})|^{2} \, d\omega \qquad (2.16)$$

and the second as

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} F(p(t)) \, \bar{F}(\phi^{n}(t) \, e^{at}) \, d\omega \qquad (2.17)$$

where the p(t) is defined in the statement of the theorem. Using the approach given in Aizerman and Gantmacher [13] an upper bound that can be obtained for (2.15) with (2.16) and (2.17) substituted into it is

$$\frac{1}{8\pi} \int_{-\infty}^{+\infty} \frac{|F(p(t))|^2}{\mathrm{Re}H(j\omega)} \, d\omega \; . \tag{2.18}$$

From the definition of $P(j\omega)$, $Q(j\omega)$, and $m(t)$ given in the statement of the theorem, an application of Parseval's theorem gives for (2.18)

$$\frac{1}{4} \int_{0}^{\infty} m^2(t)dt \; . \tag{2.19}$$

Using (2.5) on the left hand side of (2.13) together with the bound on the right hand side of (2.13) given by (2.19) results in

$$\begin{aligned} &de^{2aT_n}\,\Phi(T_n) + 2ad \int_0^{T_n} e^{2at}\,\sigma(t)\phi(t)dt - 2ad \int_0^{T_n} e^{2at}\,\Phi(t)dt \\ &+ c \int_0^{T_n} e^{at}((\delta(t) + y(t)) * [(\sigma^n(t) - \phi^n(t)/k_2)e^{at}])\phi^n(t)dt \leq \\ &\qquad (1/4) \int_0^{\infty} m^2(t)dt + d\Phi(0). \end{aligned} \tag{2.20}$$

Since the sum of the second and third integrals is non-negative, the desired bound

$$\Phi(T_n) \leq e^{-2aT_n}\left[\frac{\int_0^{\infty} m^2(t)dt}{4d} + \Phi(0)\right] \tag{2.21}$$

follows . Q.E.D.

Theorem 2.2. Let all of the conditions of theorem 2.1 hold and in addition let $\phi(\sigma)$ be an odd function. Then the assertion of theorem 2.1 holds with y(t) permitted to take on positive as well as negative values.

Proof. The only difference in the proof as compared with that of theorem 2.1 is that in place of (2.6) it is desired to show

$$\int_o^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\phi^n(t)\,dt -$$

$$\left|\int_o^{T_n} e^{at}(y(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at}))\phi^n(t)dt\right| \geq 0 \tag{2.22}$$

To show this, lemma 2 for the odd function case is applied to give

$$\left|\int_o^{T_n} e^{2a(t-\lambda)}(\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\,\phi^n(t)dt\right| \leq$$

$$\int_o^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\,\phi^n(t)dt. \tag{2.23}$$

Using (2.23) in (2.7) and (2.8) gives

$$\left|\int_o^{T_n} e^{at}\,(y(t) * ((\sigma^n(t) - \phi^n(t)/k_2)\,e^{at}))\,\phi^n(t)dt\right| \leq$$

$$\left[\sum_{j=1}^{v} |c_j|\, e^{ad_j} + {}_o\int^{\infty} e^{at}|y'(t)|dt\right]\int_o^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\phi^n(t)dt\,. \tag{2.24}$$

(2.24) shows that (2.22) holds. The remainder of the proof of the theorem is unchanged. Q.E.D.

A Simpler Bound From the Computational Standpoint

It is possible to modify (2.2) in order to obtain a simpler form for computational purposes. As the bound stands, p(t) is zero for $t > T_n$. This means that $\int_0^\infty m^2(t)dt$ has to be calculated for each value of T_n. Rather than using the transform of this truncated p(t) in the development, it is possible to use the Fourier transform of the untruncated function directly independent of T_n. While the original approach should give an improved result for small values of T_n, the latter approach definitely requires less computational effort which is important in hand calculation.

Theorem 2.3. Let the conditions of either theorem 2.1 or theorem 2.2 hold. Then the assertions of these theorems hold with p(t) replaced by

$$p(t) = e^{at}[(c + 2ad)\,\dot{\sigma}_i(t) + d\,\sigma_i(t)] + c(\sigma_i(t)e^{at} * y(t)). \tag{2.25}$$

Proof. Referring to (2.14) it is seen that the change in the definition of p(t) does not affect the value of the last three integrals on the right hand side of this equation. Also, since the new p(t) is Fourier transformable due to G(s) having poles to the left of $s = -a$ and due to (2.1) holding, it follows that the remaining steps in the proof can be carried out without any alteration. Q.E.D.

<u>Example 1</u>. Let $G(s) = \frac{1}{(s + 1)(s + 5)}$, $k_2 = 50$, and $\phi(\sigma)$ be a monotone nonlinearity. It is assumed that this system is excited by a unit impulse input. The Popov criterion shows that this system is asymptotically stable in the large. Since the Popov criterion is applicable, it is reasonable to attempt to satisfy the real part criterion with $Y(j\omega) = 0$. Since the pole of $G(s)$ closest to the origin is -1, a must be chosen less than 1. Let a be chosen arbitrarily as .5. The real part criterion c is then with c = 1

$$\text{Re}\ \left[\frac{(1 + .5d + dj\omega)}{(j\omega+.5)(j\omega+4.5)}\right] + .02 \geq 0 \ .$$

If d is chosen such that the zero of the term in brackets is located between the two poles, the real part of the first term will be non-negative and c is satisfied. Setting d = 1 gives

$$H(s) = \frac{(s + 1.5)}{(s + .5)(s + 4.5)} + .02$$

$Q(j\omega)$ obtained by factoring the reciprocal of the real part of $H(j\omega)$ is

$$\frac{7.07(s + .5)(s + 4.5)}{s^2 + 14.9s + 13.1} \ .$$

For a unit impulse input $\sigma_1(t) = .25e^{-t} - .25e^{-5t}$ and $\dot{\sigma}_1(t) = -.25e^{-t} + 1.25e^{-5t}$. Then $p(t) = e^{.5t}(2\sigma_1(t) + \dot{\sigma}_1(t)) = .25e^{-.5t} + .75e^{-4.5t}$. $\int_0^\infty m^2(t)dt$ evaluated using Parseval's theorem and tables is 1.86. Substituting this value into the bound expression gives $\Phi(t) \leq .465e^{-t}$.

In order to determine the closeness of this bound for a particular case, let $\phi(\sigma) = 50\sigma$. This choice gives $\Phi(\sigma) = 25\sigma^2$. Using the previously established bound results in $|\sigma(t)| \leq .1365e^{-.5t}$. The actual response of the system with a unit impulse input is $.1475e^{-3t} \sin 6.78t$ which has a maximum magnitude of .081 at t = .17 seconds.

C. <u>Some Considerations in Using the Theorems</u>

At first glance it might appear that the best bound would be obtained by using the largest allowed value of a. However, as the parameter a is increased, the value of the quantity multiplying the exponential term in the bound expression will generally increase since the minimum value of the real part of $H(j\omega)$ will get smaller. With bounds available for different a's, it is of course possible to combine them to get an improved overall bound by taking the smallest bound at a given time.

With regard to the allowed values for a, it has already been stated in the theorem that a must be less than the magnitude of the real part of the pole of $G(j\omega)$ closest to the j axis. By considering the linear case, it is also seen that a must lie to the right of that portion of the root locus of the system corresponding to the gain in the sector $(0, k_2)$.

Once a has been chosen, it is necessary to check the real part criterion to determine whether it is satisfied. Presumably, the asymptotic stability of the system will have been demonstrated so that a candidate for a $Y(s)$ function is available as well as a value of d/c. It is to be noted that the satisfaction of the real part condition only depends upon d/c but that the value of the bound obtained depends upon both these parameters. If the real part condition is not satisfied for this choice for all ω, the parameters can be altered and a new value of $Y(j\omega)$ selected. The required changes in the parameters and $Y(j\omega)$ should be evident from the first try.

It must always be made certain that $\int_0^\infty e^{at} \, |y(t)| dt < 1$.

A point to note is that the larger the value of a, the more difficult it is to satisfy the criterion since ad $G(j\omega - a)$ has a larger coefficient and since the area associated with $y(t)$ becomes less, implying that the maximum phase angle that can be obtained from $1 + Y(j\omega)$ is less than 90^o.

Using a computer it is possible to obtain an optimum value for the parameters c and d and for $Y(j\omega)$ by selecting these quantities to minimize the function of time or number multiplying the exponential term in the bound expression. With hand calculation techniques one would have to be satisfied with a few different trials for these quantities.

D. Case of $x(t) \neq 0$

If in order to show stability a multiplier is required which has $z(t) = F^{-1}(Z(j\omega))$ non-zero for $t \leq 0$, the bounding inequality becomes more complicated in that the value of $|\phi^n(t)|$ max, the maximum value of $|\phi^n(t)|$ in the interval $(0, T_n)$, must be used. This result is presented in the next theorem.

Theorem 2.4. For the system of figure 1 excited by initial conditions let a and b of theorem 2.1 hold and let

$$c \;\mathrm{Re}\, H(j\omega) = \mathrm{Re}[c(1 + X(j\omega) + Y(j\omega))(G(j\omega - a) + 1/k_2) + dj\omega\, G(j\omega - a) + ad\, G(j\omega - a)] \geq b > 0 \tag{2.26}$$

where b, c, and d are positive numbers, $x(t)$ and $y(t)$ are composed of delayed impulses and a piecewise continuous function that satisfy $x(t) = 0$ for $t > 0$, $y(t) = 0$ for $t < 0$, $x(t) \leq 0$ for $t < 0$, $y(t) \leq 0$ for $t > 0$. The magnitude of the piecewise continuous component of $x(t)$ is assumed to be less than $\ell \exp(ft)$ where ℓ and f are positive numbers and

$$\int_{-\infty}^{+\infty} e^{-a|t|}\, |x(t) + y(t)|\,dt < 1. \tag{2.27}$$

Then

$$\Phi(T_n) \leq e^{-2aT_n}\left[\frac{\int_0^\infty m^2(t)\,dt}{4d} + \Phi(0) + M(T_n)|\phi^n(t)|^2_{max}\right]$$

where $m(t) = F^{-1}[P(j\omega)\,Q(j\omega)]$ with

$$p(t) = e^{at}\,[(c + 2ad)\sigma_i^{\,n}(t) + d\,\dot{\sigma}_i^{\,n}(t)] +$$

$$c\,[\sigma_i^{\,n}(t)\,e^{at} * (x(t) + y(t))]^n$$

and $Q(j\omega)$ is defined by $1/\mathrm{Re}\,H(j\omega) = Q(j\omega)\,Q(-j\omega)$.

$$M(T_n) = c \int_o^{T_n} e^{at} \int_{-\infty}^{t} e^{a(t-\lambda)} \int_{t-\lambda-T_n}^{t-\lambda} |g(\varepsilon)|\,|x(\lambda)|\,d\varepsilon\,d\lambda\,dt.$$

where $g(t) = F^{-1}\,(G(j\omega))$.

<u>Proof</u>: The proof is identical with the proof of theorem 2.1 until (2.6) isreached. In place of (2.6) it is to be shown that

$$\int_o^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\,\phi^n(t)dt +$$

$$\int_o^{T_n} e^{at}[x(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at})]\,\phi^n(t)dt +$$

$$\int_o^{T_n} e^{at}\,[y(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at})]\,\phi^n(t)dt > 0\,. \qquad (2.28)$$

Let $x(t) = \sum_{i=1}^{u} a_i\,\delta(t + b_i) + x'(t)$ where $x'(t)$ is the piecewise continuous component of $x(t)$. Substituting the impulsive component of $x(t)$ in the second integral above gives

$$\sum_{i=1}^{u} a_i\,e^{ab_i} \int_o^{T_n} e^{2at}\,[\sigma^n(t + b_i) - \phi^n(t + b_i)/k_2]\,\phi^n(t)dt \qquad (2.29)$$

and substituting the piecewise continuous component $x'(t)$ into this same integral gives with a change in the order of integration

$$\int_{-\infty}^{0} x'(\lambda)\, e^{-a\lambda} \int_{0}^{T_n} e^{2at}[\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2]\phi^n(t)dt\, d\lambda. \tag{2.30}$$

Writing out $\int_{0}^{T_n} e^{2at}\, [\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2]\, \phi^n(t)dt$ as in (2.10) gives that this integral is less than or equal to

$$\int_{0}^{T_n} e^{2at}\, (\sigma_+^n(t-\lambda) - \phi_+^n(t-\lambda)/k_2)\, \phi_+^n(t)dt\, +$$

$$\int_{0}^{T_n} e^{2at}(\sigma_-^n(t-\lambda) - \phi_-^n(t-\lambda)/k_2)\, \phi_-^n(t)dt\, . \tag{2.31}$$

Applying lemma 2 for $\lambda < 0$ then gives that

$$\int_{0}^{T_n} e^{2at}\, [\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2]\, \phi^n(t)dt \leq$$

$$\int_{0}^{T_n} e^{2at}\, [\sigma^n(t) - \phi^n(t)/k_2]\, \phi^n(t)dt\, . \tag{2.32}$$

A lower bound on the second and third integrals of (2.28) is then (2.32) times

$$[\sum_{i=1}^{u} a_i\, e^{ab_i} + \sum_{j=1}^{v} c_i\, e^{ad_i} + \int_{-\infty}^{+\infty} (x'(t)\, e^{-at} + y'(t)e^{+at})]dt$$

which shows that (2.28) holds.

Next, let the term

$$c \int_{o}^{T_n} e^{at} \, (x(t) * ((\sigma^n(t) - \phi^n(t)/k_2)e^{at})) \; \phi^n(t)dt$$

be added to (2.13) and let the substitution be made as before. A modification is required in the replacement of $\sigma_\phi^{\ n}(t)$ by $\sigma_\phi^{\ n*}(t)$ for the added integral on the right hand side of (2.13). For this integral it is necessary to take into account the difference between these two functions due to x(t)'s being non-zero for $t < 0$. Let $\sigma_\phi^{\ n*}(t) = \sigma_\phi^{\ n}(t) + \sigma_\phi^{\ d}(t)$. Substituting for $\sigma_\phi^{\ n}(t)$ according to this expression then gives the following two integrals to be added to (2.14)

$$c \int_{-\infty}^{+\infty} e^{at}(x(t) * [(\sigma_\phi^{\ n}(t) - \phi^n(t)/k_2)e^{at}]) \; \phi^n(t)dt$$

$$- c \int_{-\infty}^{+\infty} e^{at}(x(t) * (\sigma_\phi^{\ d}(t)e^{at})) \; \phi^n(t)dt \; . \tag{2.33}$$

An added term involving the initial condition expression is

$$c \int_{-\infty}^{+\infty} e^{at} \, [x(t) * \sigma_i^{\ n}(t)e^{at}] \; \phi^n(t)dt \; . \tag{2.34}$$

As in the proof of the corresponding stability theorem, the magnitude of the integral involving $\sigma_\phi^{\ d}(t)$ can be bounded in terms of $|\phi^n(t)|_{max}$. Using the definition of $|\phi^n(t)|_{max}$ and taking absolute magnitudes gives

$$c \int_{-\infty}^{+\infty} e^{at}(x(t) * (\sigma_{\phi}{}^{d}(t)\ e^{at}))\ \phi^{n}(t)dt \leq$$

$$c\ |\phi^{n}(t)|^{2}_{max} \int_{o}^{T_n} e^{at} \int_{-\infty}^{t} e^{a(t-\lambda)} \int_{t-\lambda-T_n}^{t-\lambda} |g(\varepsilon)||x(\lambda)|d\varepsilon d\lambda dt$$

$$= M(T_n)\ |\phi^{n}(t)|^{2}_{max}\ . \tag{2.35}$$

Repeating the steps in (2.15) through (2.19) then gives for (2.20)

$$de^{2aT_n}\ \Phi(T_n) + 2ad \int_{o}^{T_n} e^{2at}\ \sigma(t)\phi(t)dt - 2ad \int_{o}^{T_n} e^{2at}\ \Phi(t)dt$$

$$+ c \int_{o}^{T_n} e^{at}\ ((\delta(t) + x(t) + y(t)) * [(\sigma^{n}(t) - \phi^{n}(t)/k_2)e^{at}])\ \phi^{n}(t)dt$$

$$\leq \frac{1}{4} \int_{o}^{\infty} m^{2}(t)dt + d\Phi(0) + M(T_n)\ |\phi^{n}(t)|^{2}_{max}\ . \tag{2.36}$$

Then (2.21) becomes

$$\Phi(T_n) \leq e^{-2aT_n}\ [\frac{\int_{o}^{\infty} m^{2}(t)dt}{4d} + \Phi(0) + \frac{M(T_n)}{d}|\phi^{n}(t)|^{2}_{max}]. \tag{2.37}$$

Q.E.D.

Theorem 2.4 can be applied in the case where $\phi(\sigma)$ is an odd monotone function with $x(t)$ and $y(t)$ being less restricted. The proof is similar to that of theorem 2.1 so it will not be repeated here.

<u>Theorem 2.5</u>. Let all of the conditions of theorem 2.4 hold and in addition let $\phi(\sigma)$ be an odd function. Then the assertion of theorem 2.4 holds with $x(t)$ and $y(t)$ permitted to take on positive as well as negative values

Although $M(T_n)$ is independent of system excitation as developed in the proof of the theorem, this is not the case for $|\phi^n(t)|_{max}$. A value must be obtained for this quantity before the bound can be applied. The simplest way to find this quantity is by using theorems 2.4 or 2.5 with a = 0. T_n is chosen as that value of time at which $|\phi^n(t)|_{max}$ occurs. Then by using the fact that $\Phi(T_n)$ approaches infinity more rapidly than $|\phi(\sigma)|^2$, a bound can be obtained on $|\phi|$ by finding the value of this variable above which the bounding inequality does not hold.

E. A Different Bound

The bound (2.2) given by theorem 2.1 as well as the other bounds obtained thus far depend upon the square of the initial condition excitation. As long as $\phi(\sigma)$ is in its linear range, a reasonable bound is obtained for σ. To see this, let $\Phi(\sigma) = c_1\sigma^2$ where c_1 is a positive number. In the calculation of the bound for σ a square root must be taken and σ is then effectively bounded by a linear function of the initial conditions. On the other hand if $\phi(\sigma)$ is in a saturation region, $\Phi(\sigma) = c_2|\sigma| + c_3$, resulting in the bound depending upon the square of the initial conditions. To try to get a better estimate in this saturation case, the approach employed by Lefschetz [11] will be used rather than the "completing the square" approach given in Aizerman and Gantmacher [13] that has been utilized thus far. The Lefschetz approach yields a bound dependent upon the magnitude of the initial conditions.

Theorem 2.6 Let all of the conditions of either theorems 2.1 or 2.2 hold. Then another bound on $\Phi(T_n)$ is

$$\Phi(T_n) \leq e^{-2aT_n} \left[\frac{|\phi^n(t)|_{max} \int_0^\infty |p(t)|dt}{d} + \Phi(0) \right] \tag{2.38}$$

where

$$p(t) = e^{2at}[(c + 2ad)\ \sigma_i{}^n(t) + d\ \dot{\sigma}_i{}^n(t)] + c\ e^{at}[\sigma_i{}^n(t)e^{at} * (y(t))]^n \tag{2.39}$$

Proof. The proof is unchanged until (2.15) is reached. At this point, since (2.16) is negative, it can be dropped and the second integral (2.17) retained. Then, the left hand side of (2.20) is less than or equal to the magnitude of (2.17) written in time domain form which is

$$c \int_{-\infty}^{+\infty} e^{at}((\delta(t) + y(t)) * [\sigma_i{}^n(t)e^{at}])^n\ \phi^n(t)dt +$$

$$d \int_{-\infty}^{+\infty} e^{2at}\ \dot{\sigma}_i{}^n(t)\ \phi^n(t)dt + 2da \int_{-\infty}^{+\infty} e^{2at}\ \sigma_i{}^n(t)\ \phi^n(t)dt. \tag{2.40}$$

The magnitude of this integral is less than or equal to

$$|\phi^n(t)|_{max} \int_{-\infty}^{+\infty} |p(t)|dt\ . \tag{2.41}$$

where $p(t)$ is defined above. With the exception of the use of the new bound, the remainder of the proof is unchanged. Q.E.D.

In a similar way theorems 2.4 and 2.5 can be restated using this new bound. The modification in the proof is identical to that given for theorem 2.6.

Theorem 2.7. Let all of the conditions of either theorems 2.4 or 2.5 hold. Then another bound on $\Phi(T_n)$ is

$$\Phi(T_n) \le e^{-2aT_n} \left[\frac{|\phi^n(t)|_{max} \int_0^\infty |p(t)|dt}{d} + \frac{M(T_n)}{d} |\phi^n(t)|^2_{max} + \Phi(0) \right] \tag{2.42}$$

where

$$p(t) = e^{2at} [(c + 2ad)\, \sigma_i^{\,n}(t) + d\, \dot{\sigma}_i^{\,n}(t)] +$$

$$c\, e^{at}[\sigma_i^{\,n}(t)e^{at} * (x(t) + y(t))]^n \tag{2.43}$$

and $M(T_n)$ is defined in the statement of theorem 2.4.

Example 2. Consider the same problem as that of example 1 and let the nonlinear characteristic be a saturation function defined by $\phi(\sigma) = 50\sigma$ for $0 \le |\sigma| \le .02K$ and $\phi(\sigma) = \pm K$ for $.02K \le |\sigma| < \infty$ with the + sign applying for positive values of σ and the - sign for negative values. Using (2.39) and the previously computed values of $\sigma_i(t)$ and $\dot{\sigma}_i(t)$ gives $p(t) = .25 + .75e^{-4t}$. The bound is then $\Phi(T_n) \le K\,(.25T_n\, e^{-T_n} - .1875e^{-5T_n} + .1875e^{-T_n})$ with $|\phi^n(t)|_{max} = K$. The bounds for σ are then

$$|\sigma| \le .25\, T_n\, e^{-T_n} - .1875e^{-5T_n} + .1875e^{-T_n} + .01K, |\sigma| \ge .02K$$

$$\sigma^2 \le .04K(.25T_n e^{-T_n} - .1875e^{-5T_n} + .1875e^{-T_n}), |\sigma| \le .02K .$$

Plots of this bound (called the L bound) and of the bound computed in example 1 (called the AG bound) are plotted in figures 7-10 for various values of the saturation level K. The smaller the value of K, the better the results of the L bound as compared with the AG bound.

F. A Response Bound With an External Input Applied

The introduction of the e^{at} multiplier for $\Phi(T_n)$ allows a bound to be obtained for the response of the system with certain external inputs applied. Theoretically, it is only necessary to make certain that the input is such that piecewise continuity and Fourier transformability are guaranteed for certain pertinent functions. From the practical standpoint some difficulty may be encountered in finding a bound for $|\phi^n(t)|_{max}$ in theorems 2.4, 2.5, 2.6, and 2.7. If $\int_{-\infty}^{\infty} |p(t)|dt$ is bounded for a = 0, a bound can be computed as discussed previously; if this integral is not bounded, it is necessary to calculate a time varying bound for $|\phi^n(t)|^2_{max}$ using the theorems with a = 0 and choosing $|\phi^n(t)|^2_{max}$ as occurring at $t = T_n$ as the worst case. Since $|\phi^n(t)|_{max}$ does not appear in theorems 2.1 and 2.2, these theorems can be applied with no change in the computation procedure. Examples of possible inputs include a sinusoidal function, a ramp function and an exponential function. This discussion is summarized in the following theorem.

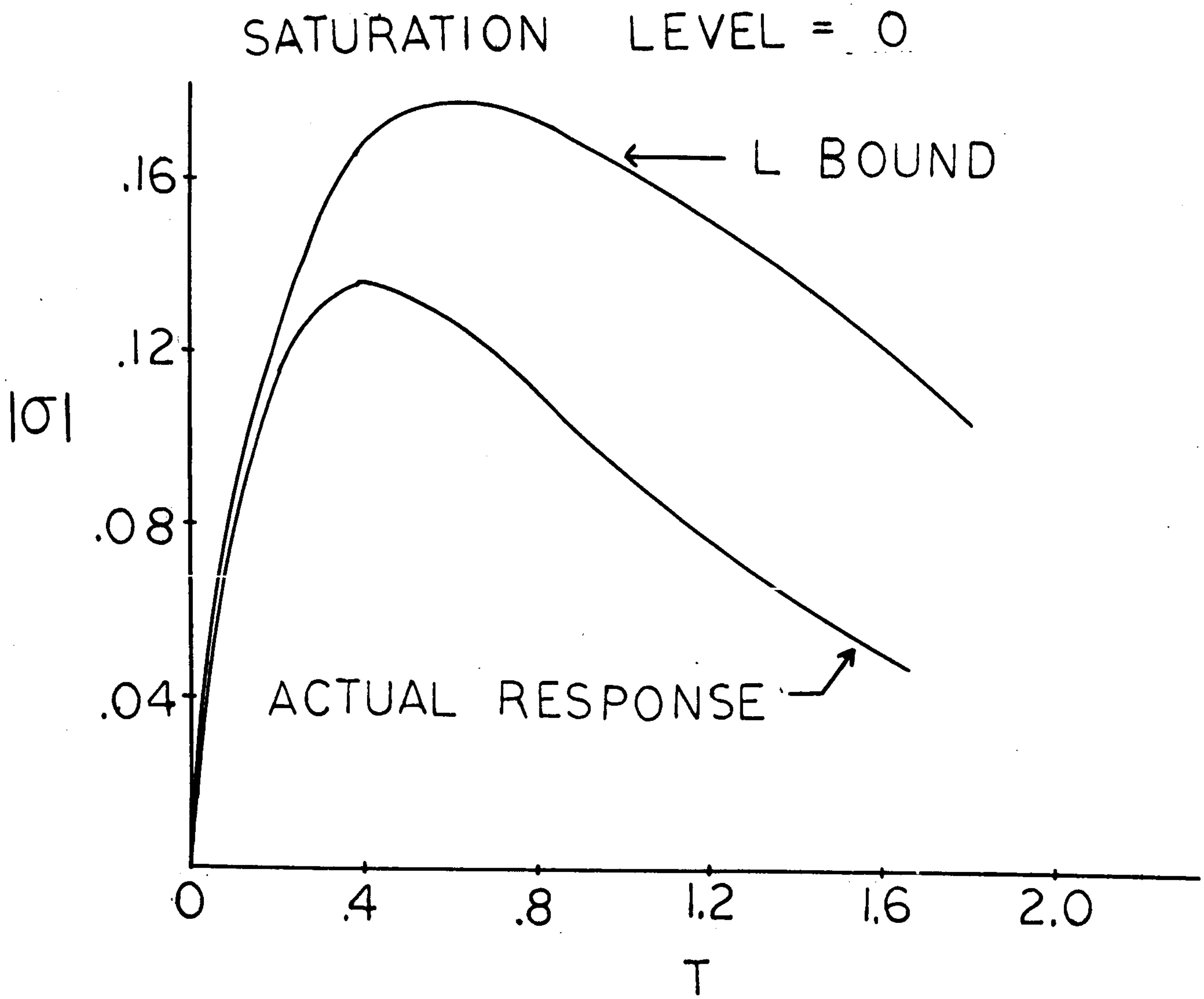

Figure 7. Bound on σ for the saturation level k = 0.

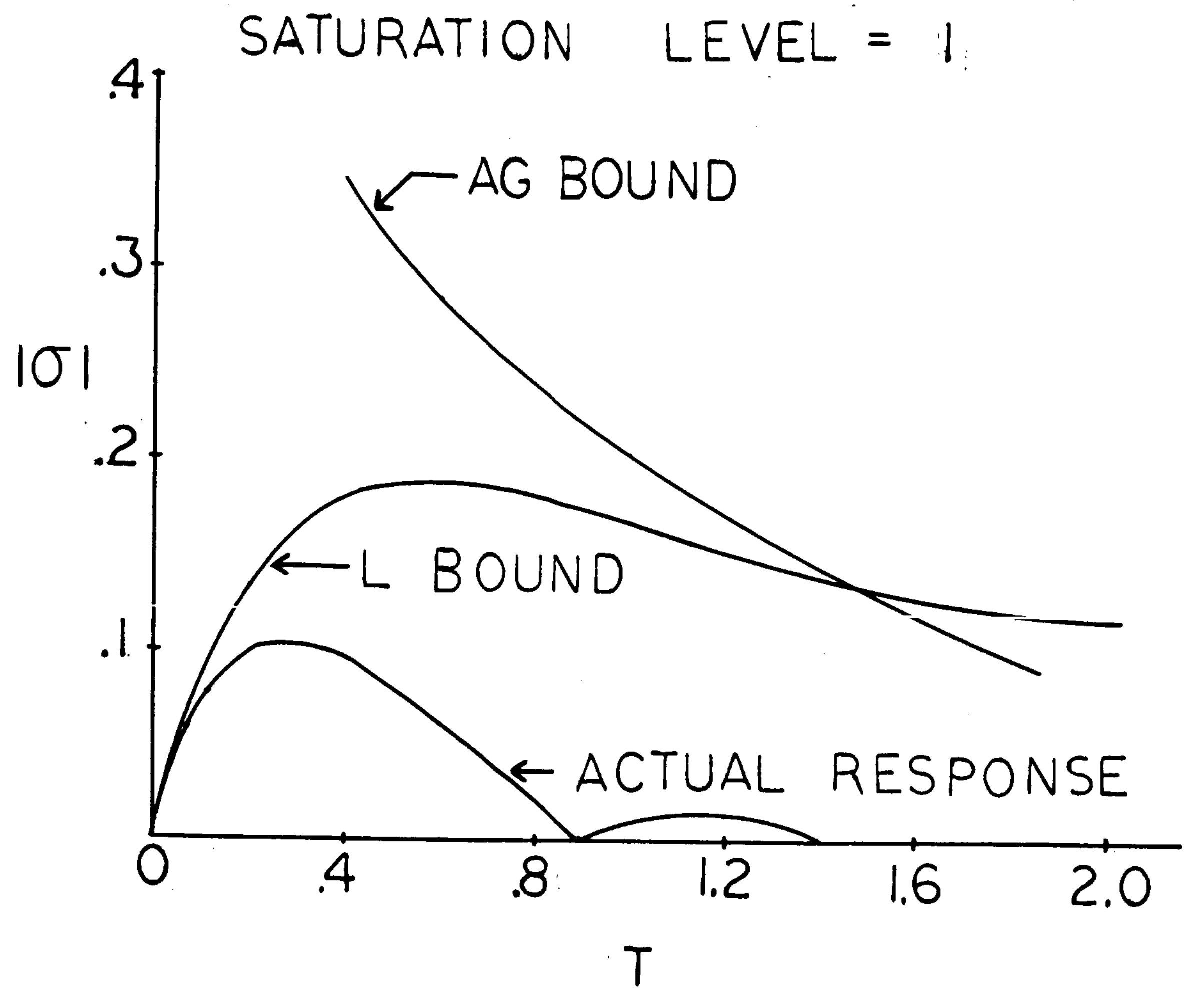

Figure 8. Bounds on σ for the saturation level k = 1.

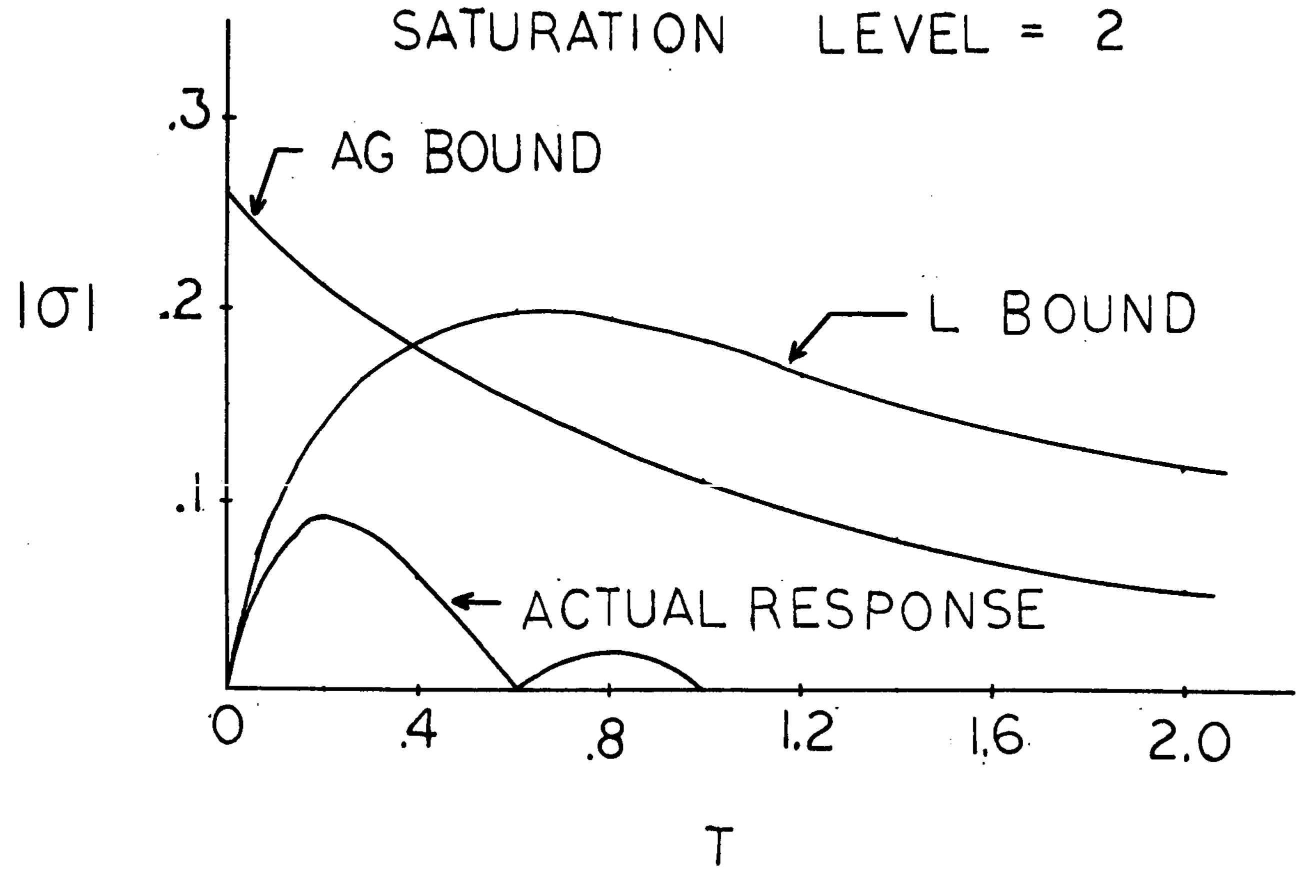

Figure 9. Bounds on σ for the saturation level k = 2.

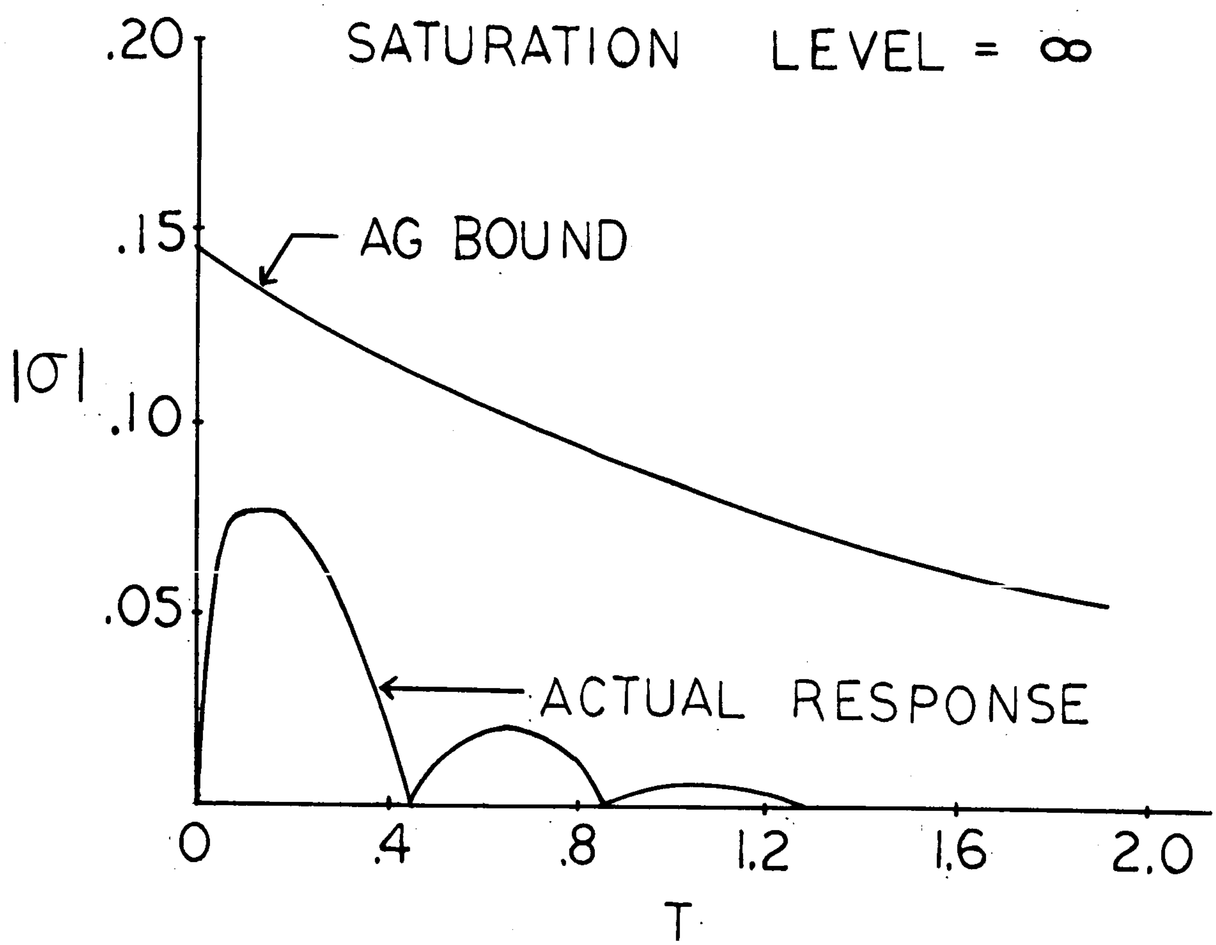

Figure 10. Bounds on σ for the saturation level k = ∞ (linear case).

Theorem 2.8. Let the conditions of either theorem 2.1, 2.2, 2.4, 2.5, 2.6, or 2.7 hold. If the input to the system is such that $\sigma^n(t)$, $\dot{\sigma}^n(t)$, $\phi^n(t)$, $\sigma_r^{\ n}(t)$, and $\dot{\sigma}_r^{\ n}(t)$ are Fourier transformable, the assertions of these theorems hold with $\sigma_i^{\ n}(t)$ and $\dot{\sigma}_i^{\ n}(t)$ replaced by $\sigma_i^{\ n}(t) + \sigma_r^{\ n}(t)$ and $\dot{\sigma}_i^{\ n}(t) + \dot{\sigma}_r^{\ n}(t)$ respectively. $\sigma_r^{\ n}(t)$ and $\dot{\sigma}_r^{\ n}(t)$ are equal to those components of $\sigma(t)$ and $\dot{\sigma}(t)$, respectively, due to the direct action of the input (the input acting through G(s)) in $(0, T_n)$ and zero outside this interval.

Example 3. Let $G(s) = 1/(s + 1)$, $k_2 = 10$, the nonlinearity be monotone, and the excitation be an input of sin t with the initial conditions zero. This G(s) is sufficiently simple that theorem 1 can be applied with $y(t) = 0$.

$$\text{Re } H(j\omega) = \frac{\text{Re } (c + ad + dj\omega)}{j\omega - a + 1} + .1 .$$

Set a = .25, c = 1, and d = 2. This then gives $\text{Re}H(j\omega) = 2.1$. $\sigma_r(t) = .5e^{-t} + .707 \cos(t - 135^\circ)$, $\dot{\sigma}_r(t) = -.5e^{-t} - .707 \sin(t - 135^\circ)$, and $p(t) = 2e^{.25t} \sin t$. Using these quantities then gives as the bound

$$\Phi(T_n) \leq .238 - .014 \cos 2T_n - .0561 \sin 2T_n - .2235e^{-.5T_n} \leq .2959.$$

For the special case $\phi(\sigma) = 10\sigma$, using the above bound gives $\sigma(T_n) \leq .243$. The actual response is $\sigma(t) = .0082e^{-11t} + .0905 \cos(t - 95.2^\circ)$.

Example 4. Let the system be the same as in example 3 but let the input be a unit ramp rather than a sinusoidal input. $\sigma_r(t) = t-1+e^{-t}$, $\dot{\sigma}_r(t) = 1 - e^{-t}$, $p(t) = 2e^{.25t}t$. Using these values gives

$$\Phi(T_n) \leq 1.91(.25\, T_n^2 - T_n + 2) - 3.82e^{-.5T_n}$$

from which it is seen that the bound approached for large T_n is $.4775\, T_n^2$. With $\phi(\sigma) = 10\sigma$, this gives as a bound for large T_n $|\sigma| \leq .309\, T_n$. The actual response for large values of T_n is $\sigma = .0909\, T_n$.

For both of these examples by referring to [6] - [9] and treating the inputs as being zero outside $(0, T_n)$, it can be shown that the conditions of the theorem are satisfied.

As was pointed out in the introduction, the application of this theorem can show Liapunov stability with certain inputs applied. The case of example 3 with the sinusoidal input applied illustrates this point.

G. Modification For the Case of Poles to The Left of the Line s = -a

In the case of a system in which a lag compensator has been incorporated in order to increase the gain of the system at low frequencies, the significant portions of the response are usually characterized by one time constant while another time constant due to the lag compensator characterizes the response for large values of time. In the theorems discussed thus far, it has been assumed that a is less than the magnitude of the real part of the pole closest to the origin. Therefore, these theorems would only be able to yield a bound that would be realistic for large t. The theorem below allows the calculation of a bound that should give good results for the significant portions of the response of these systems. The approach used is basically one in which the given G(s) is replaced by another transfer function equal to g(t) in $(0, T_n)$ but different from g(t) outside this interval. This modification allows the original theorems to be applied to give a bound valid in the time interval $(0, T_n)$.

Theorem 2.9. Let

$$G(s) = G_1(s) + \sum_{i=1}^{n} \frac{a_i}{s + b_i}$$

$$s\, G(s) = G_2(s) + \sum_{i=1}^{n} \frac{c_i}{s + d_i}$$

where $a > b_i$ but less than the magnitudes of the real parts of the poles of $G_1(s)$ and $G_2(s)$. Then if conditions a and b are satisfied and the modified c given below is also satisfied

$$\text{Re } H(j\omega) = \text{Re}[c(1 + Y(j\omega) + X(j\omega))\,(G_A(j\omega - a) + 1/k_2)$$

$$+ d\, G_B(j\omega - a) + 2ad\, G_A(j\omega - a)] \geq \delta > 0$$

where $X(j\omega)$ may be zero, and

$$G_A(j\omega - a) = G_1(j\omega - a) + \sum_{i=1}^{n} \frac{a_i[1 - e^{(a-b_i)T_n} e^{-j\omega T_n}]}{j\omega - a + b_i}$$

$$G_B(j\omega - a) = G_2(j\omega - a) + \sum_{i=1}^{n} \frac{c_i[1 - e^{(a-b_i)T_n} e^{-j\omega T_n}]}{j\omega - a + b_i}$$

the assertions of theorems 2.1, 2.2, and 2.6 hold without any changes and the assertions of theorems 2.4, 2.5, and 2.7 hold with the $g(\varepsilon)$ used in the definition of $M(T_n)$ replaced by $g_A(\varepsilon) = F^{-1}\,[G_A(j\omega)]$.

<u>Proof</u>. The only change required in the proof of the theorems is in the step just before the application of Parseval's theorem by which the time domain integrals are converted to frequency domain integrals. $\sigma_\phi^{n*}(t)$ and $\dot{\sigma}_\phi^{n*}(t)$ are redefined as $\sigma_\phi^{n*}(t) = -F^{-1}[G_A(j\omega)\ F(\phi^n(t))]$ and $\dot{\sigma}_\phi^{n*}(t) = -F^{-1}[G_B(j\omega)\ F(\phi^n(t))]$. If $x(t) = 0$, these changes do not alter the values of the integrals in which they appear since these two time functions are equal to $\sigma(t)$ and $\dot{\sigma}(t)$, respectively, in $(0, T_n)$. For $x(t) \neq 0$, the substitutions result in a different value for $\sigma_\phi^d(t)$ but the same steps in the proof are applicable with $g(\varepsilon)$ being replaced by $g_A(\varepsilon)$ in the definition of $M(T_n)$. The reason for the changes is that with the original definitions, $\sigma_\phi^{n*}(t)$ and $\dot{\sigma}_\phi^{n*}(t)$ when multiplied by e^{at} were not Fourier transformable. The new definitions result in Fourier transformable functions when multiplied by the exponential. The other steps in the proof are unchanged.

Example 5. Let $G(s) = -1.005/(s + 2) + 1/(s + 1) + .005/(s + .1)$ for a system with a monotone nonlinearity and a gain $k_2 = 10$. Then $s\,G(s) = 2.010/(s + 2) - 1/(s + 1) - .0005/(s + .1)$. Let $a = .5 = T_n$. This $G_A(s)$ is sufficiently simple that theorem 2.1 can be applied with $Y(s) = 0$, $c = 1$, and $d = 1$. The real part criterion is then

$$\mathrm{Re}\,[2\,G_A(j\omega - a) + G_B(j\omega - a) + .1] =$$

$$1/(j\omega + .5) + .0095(1 - e^{+.4T_n}\,e^{-j\omega T_n})/(j\omega - .4) + .1.$$

The maximum magnitude of the second term on the right hand side is .053. Therefore, $\mathrm{Re}\,H(j\omega) \geq .047$. For convenience in the calculation of the lower bound, this number will be used rather than the actual function of frequency. $p(t) = e^{.5t}[2\sigma_i^{\,n}(t) + \dot{\sigma}_i^{\,n}(t)] = e^{-.5t} + .0095e^{+.4t}$ for an impulse input. The bound is then

$$\Phi(T_n) \leq 5.32\,[1.19e^{-T_n} + .000113\,e^{-.2T_n} - e^{-2T_n} - .19\,e^{-1.1T_n}]$$

for $T_n \leq .5$. For $T_n \geq .5$, the bound given by the original theorems can be used with $a < .1$.

H. A Result for the Linear Case

If $\phi(\sigma)$ is a linear function or a nonlinear function in its linear range, it is possible to get an improved result for the frequency domain condition c. To see this, let $\phi(\sigma) = K\sigma$ where $0 < K < \infty$. Then (2.4) can be replaced by

$$a \int_o^{T_n} e^{2at}\,\sigma(t)\,\phi(t)dt - 2a \int_o^{T_n} e^{2at}\,\Phi(t)dt \geq 0$$

since $\sigma(t)\phi(t) = K\sigma^2(t)$ and $\Phi(t) = K\sigma^2(t)/2$. Also, for the linear case

$$\int_o^{T_n} e^{at}\, e^{a(t-\lambda)}\, (\sigma^n(t-\lambda) - \phi^n(t-\lambda)/k_2)\phi^n(t)dt =$$

$$K(1 - K/k_2) \int_o^{T_n} e^{at}\,\sigma^n(t) e^{a(t-\lambda)}\sigma^n(t-\lambda)dt \leq$$

$$.5K(1 - K/k_2)\; [\int_o^{T_n} e^{2at}\sigma^{n2}(t)dt + \int_o^{T_n} e^{2a(t-\lambda)}\sigma^{n2}(t-\lambda)dt] \leq$$

$$K(1 - K/k_2) \int_o^{T_n} e^{2at}\sigma^{n2}(t)dt = \int_o^{T_n} e^{2at}(\sigma^n(t) - \phi^n(t)/k_2)\,\phi^n(t)dt$$

which means that the integral magnitude condition can be relaxed. Combining these two results gives for the frequency domain condition c

$$\mathrm{Re}\, H(j\omega) =$$

$$\mathrm{Re}[(1 + dj\omega + Y(j\omega) + X(j\omega))(G(j\omega - a) + 1/k_2)] \geq \delta > 0$$

where $\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)dt < 1.$

Because of this improved condition, it is possible to choose larger values of the parameter a for the linear case than for the nonlinear case. This suggests the following approach. When $|\sigma(t)|$ is such that the system is in its nonlinear region, one of the bounds already discussed can be claculated. When according to this bound the system is in and remains in the linear region for all succeeding values of t, say $t \geq T^1$, an improved bound is calculated using the real part criterion given above. In applying the theorem this second time, a value is immediately available for $\Phi(T^1)$. However, since $\sigma_i(t)$ and $\dot{\sigma}_i(t)$ are not known for this second application of the theorem, bounds for these two quantities must be calculated using the bound on $\phi(\sigma)$ determined in the first application of the theorem.

I. Conclusion

This chapter has presented a number of different results for bounds on the response of the single nonlinearity time invariant system. The usefulness of these bounds appears to be in two applications. First, it is possible to develop an approach for carrying out an analytical design for a nonlinear system. If the system is excited by initial conditions or by an impulse or step input which can be converted to equivalent initial condition inputs, the theorems given can be used to calculate a bound on $|\sigma(t)|$. Since the desired equilibrium state for the excitation under discussion is the origin, it is possible to obtain a satisfactory design for the response time of the system by adjusting the parameters of the system or by adding a compensator such that the bound on the system output meets the system specifications. Secondly, if a bounded time varying input is applied to the system, it is possible to show Liapunov stability by applying the bounding theorems.

Therefore, the bounding theorems give sufficient conditions for Liapunov stability with a bounded input applied, provided that no common factors of G(s) in the right half s plane or on the $j\omega$ axis have been cancelled.

J. Appendix

Lemma 1. Let $f_a(t)$ and $f_b(t)$ be two continuous functions of t that are zero outside the interval $(0, n\Delta t)$ where n is a positive integer and Δt is a positive number, $f_a(t)\, f_b(t) \geq 0$, $f_a(t) = h(f_b(t))$ where h is a piecewise continuous monotone increasing function of its argument, then if either both $f_a(t)$ and $f_b(t)$ are always non-positive, or non-negative or if h is an odd function with $h(0) = 0$,

$$\sum_{k=o}^{n} |f_a(k\Delta t)\; f_b(k\Delta t - \lambda)| \leq \sum_{k=o}^{n} f_a(k\Delta t)\; f_b(k\Delta t)$$

where λ is a real number such that $|\lambda|/\Delta t$ is an integer.

Proof. The proof of this lemma follows from the proof of the lemma given at the end of chapter 1 in which this result is obtained as an intermediate step.

Lemma 2. Let $f_a(t)$ and $f_b(t)$ be two continuous functions of time that are zero outside the interval $(0, T_n)$ where T_n is a positive number, $f_a(t)\, f_b(t) \geq 0$, $f_a(t) = h(f_b(t))$ where h is a piecewise continuous monotone increasing function of its argument, then if either both $f_a(t)$ and $f_b(t)$ are always non-positive or non-negative or if h is an odd function with $h(0) = 0$,

$$\left| \int_o^{T_n} e^{2a(t-\lambda)} f_a(t) f_b(t-\lambda) dt \right| \leq \int_o^{T_n} e^{2at} f_a(t) f_b(t) dt, \quad \lambda > 0$$

and

$$\left| \int_o^{T_n} e^{2at} f_a(t) f_b(t-\lambda) dt \right| \leq \int_o^{T_n} e^{2at} f_a(t) f_b(t) dt, \quad \lambda < 0.$$

Proof. Let Δt be chosen such that $|\lambda|/\Delta t$ is a positive integer and n is the largest integer less than or equal to $T_n/\Delta t$. It is assumed that $|\lambda| < T_n$ for if $|\lambda| \geq T_n$, the assertion of the lemma follows at once. With $\lambda > 0$, let the two summations

$$\sum_{k=o}^{n} |f_a(k\Delta t)\ f_b(k\Delta t - \lambda)|\ e^{2a(k\Delta t - \lambda)}\ \Delta t \tag{A1}$$

and

$$\sum_{k=o}^{n} f_a(k\Delta t)\ f_b(k\Delta t) e^{2a(k\Delta t)}\ \Delta t \tag{A2}$$

be formed. (A1) divided by Δt may be rewritten as

$$[|f_a(\lambda)\ f_b(0)| + |f_a(\lambda+\Delta t)\ f_b(\Delta t)| + |f_a(\lambda+ 2\Delta t)\ f_b(2\Delta t)| + \ldots$$

$$+ |f_a(T_n-\Delta t)\ f_b(T_n- \lambda-\Delta t)| + |f_a(T_n)\ f_b(T_n-\lambda)|]$$

$$+ (e^{2a\Delta t} - 1)\ [|f_a(\lambda+\Delta t)\ f_b(\Delta t)| + |f_a(\lambda+ 2\Delta t)\ f_b(2\Delta t)| + \ldots$$

$$|f_a((n-1)\Delta t)\ f_b((n-1)\Delta t - \lambda)| + |f_a(n\Delta t)\ f_b(n\Delta t-\lambda)|]$$

$$+ (e^{4a\Delta t} - e^{2a\Delta t})[|f_a(\lambda+ 2\Delta t)\ f_b(2\Delta t)| + |f_a(\lambda+3\Delta t)\ f_b(3\Delta t)| + \ldots$$

$$|f_a((n-1)\Delta t)\ f_b((n-1)\Delta t-\lambda)| + |f_a(n\Delta t)\ f_b(n\Delta t-\lambda)|]$$

$$+ \ldots$$

$$+ (e^{2a(\lambda-\Delta t)} - e^{2a(\lambda-2\Delta t)})\ [|f_a((n-1)\Delta t)\ f_b((n-1)\Delta t-\lambda)| +$$

$$|f_a(n\Delta t)\ f_b(n\Delta t-\lambda)|]$$

$$+ (e^{2a\lambda} - e^{2a(\lambda-\Delta t)})\ |f_a(n\Delta t)\ f_b(n\Delta t-\lambda)|. \tag{A3}$$

Similarly, (A2) divided by Δt may be rewritten as

$$[f_a(0)\ f_b(0) + f_a(\Delta t)\ f_b(\Delta t) + f_a(2\Delta t)\ f_b(2\Delta t) + \ldots$$

$$+ f_a((n-1)\Delta t)\ f_b((n-1)\Delta t) + f_a(n\Delta t)\ f_b(n\Delta t)]$$

$$+ (e^{2a\Delta t} - 1)[f_a(\Delta t)\ f_b(\Delta t) + f_a(2\Delta t)\ f_b(2\Delta t) + \ldots + f_a(n\Delta t)\ f_b(n\Delta t)]$$

$$+ (e^{4a\Delta t} - e^{2a\Delta t}) \; [f_a(2\Delta t) \; f_b(2\Delta t) + f_a(3\Delta t) \; f_b(3\Delta t) + \ldots \; f_a(n\Delta t) \; f_b(n\Delta t)]$$

$$+ \ldots$$

$$+ (e^{2a\lambda} - e^{2a(\lambda-\Delta t)})[f_a(\lambda) f_b(\lambda) + f_a(\lambda+\Delta t) f_b(\lambda+\Delta t) + \ldots \; f_a(n\Delta t) \; f_b(n\Delta t)]$$

$$+ \ldots$$

$$+ (e^{2an\Delta t} - e^{2a(n-1)\Delta t}) \; [f_a(n\Delta t) \; f_b(n\Delta t)] \; . \qquad \text{(A4)}$$

Comparing the terms in (A3) and (A4) having the same exponential multiplier and using lemma 1 on the terms of (A3), it follows that (A3) is less than or equal to (A4). Since

$$\left| \int_o^{T_n} f_a(t) \; f_b(t-\lambda) \; e^{2a(t-\lambda)} \; dt - \sum_{k=o}^{n} \; f_a(k\Delta t) \; f_h(k\Delta t-\lambda)\Delta t \right|$$

$$< \varepsilon(\Delta t)$$

where $\varepsilon(\Delta t)$ is a positive number whose value depends upon Δt, taking the limit as $\Delta t \rightarrow 0$ gives

$$\left| \int_o^{T_n} f_a(t) \; f_b(t-\lambda) \; e^{2a(t-\lambda)} dt \right| \leq \int_o^{T_n} \; f_a(t) \; f_b(t) \; e^{2at} \; dt.$$

which is one half of the lemma.

With $\lambda < 0$ the summation

$$\sum_{k=o}^{n} \; |f_a(k\Delta t) \; f_b(k\Delta t-\lambda)| e^{2ak\Delta t} \; \Delta t \qquad \text{(A5)}$$

is formed and rewritten as

$$[|f_a(0)\ f_b(-\lambda)| + |f_a(\Delta t)\ f_b(\Delta t-\lambda)| + |f_a(2\Delta t)\ f_b(2\Delta t-\lambda)| + \ldots$$

$$|f_a(n\Delta t+\lambda)\ f_b(n\Delta t)|]$$

$$+ (e^{2a\Delta t} - 1)[|f_a(\Delta t)\ f_b(\Delta t-\lambda)| + |f_a(2\Delta t)\ f_b(2\Delta t-\lambda)| + \ldots$$

$$|f_a(n\Delta t+\lambda)\ f_b(n\Delta t)|]$$

$$+ (e^{4a\Delta t} - e^{2a\Delta t})\ [|f_a(2\Delta t)\ f_b(2\Delta t-\lambda)| + |f_a(3\Delta t)\ f_b(3\Delta t-\lambda)| + \ldots$$

$$|f_a(n\Delta t+\lambda)\ f_b(n\Delta t)|]$$

$$+ \ldots$$

$$+ (e^{2a(n\Delta t+\lambda)} - e^{2a((n-1)\Delta t+\lambda)})\ [|f_a(n\Delta t+\lambda) f_b(n\Delta t)|]\ . \qquad \text{(A6)}$$

Repeating the foregoing reasoning with (A6) replacing (A3) gives

$$\left| \int_o^{T_n} e^{2at}\ f_a(t)\ f_b(t-\lambda)dt \right| \leq \int_o^{T_n} e^{2at}\ f_a(t)\ f_b(t)dt\ .$$

Q.E.D.

V CHAPTER III. SYSTEM WITH A TIME VARYING NONLINEARITY, SAMPLED DATA SYSTEMS AND SYSTEMS WITH MULTIPLE NONLINEARITIES

A. Time Varying Nonlinearity

The theorem given below is a modification of theorem 1.2 with the modification added to take into account the $\int_o^{T_n} \dot{\sigma}(t)\phi(t)dt$ term no longer being an exact integral. There are a number of ways in which this could be done; the approach used has the merit that it is not necessary to take into account the rate at which the nonlinearity changes with time. Therefore, this theorem appears to be the most generally applicable one that could be developed.

Pertinent references include the works by Sandberg [18] and Rekasius and Rowland [19]. The criteria which are developed in these references do not include anything as general as the Z(s) multiplier used in theorem 3.1.

Theorem 3.1. For the system of figure 1 with ϕ being a time varying nonlinearity let the following conditions hold:

a. $A\ \phi_m(\sigma)\sigma \leq \phi(\sigma,t)\sigma \leq B\ \phi_m(\sigma)\sigma$ where A and B are real numbers satisfying $0 < A \leq 1$ and $1 \leq B < \infty$, $\phi(0,t) = \phi_m(0) = 0$, $\sigma\ \phi(\sigma,t) < k\ \sigma^2$ where $k > 0$ and $\sigma\ \phi_m(\sigma) > 0$ for $\sigma \neq 0$, $d\phi(\sigma,t)/d\sigma$ is a continuous function of σ, $\phi_m(\sigma)$ is a continuous monotone increasing function of σ having an odd part $\phi_{mo}(\sigma)$ that satisfies $|\phi_m(\sigma)| \leq C|\phi_{mo}(\sigma)|$ and $|\phi_{mo}(\sigma)| \leq D|\phi_m(\sigma)|$.

b. Conditions b and c of theorem 1.1.

Then a sufficient condition for asymptotic stability in the large is that

$$\mathrm{Re}[Z(j\omega)\ G(j\omega) + E(G(j\omega) + 1/k) - \alpha(\tfrac{B-A}{2A})\ (k^2 + \omega^2)\ |G(j\omega)|^2] \geq \delta > 0 \quad (3.1)$$

for all real ω where E is a non-negative number, δ is a positive number, and

$$Z(j\omega) = 1 + \alpha j\omega + X(j\omega) + Y(j\omega) \tag{3.2}$$

and

$$\frac{BCD}{A} \left[\int_{-\infty}^{+\infty} (x'^{+}(t) + y'^{+}(t))dt + \sum a_i^{+} + \sum c_i^{+} \right] -$$

$$\frac{B}{A} \left[\int_{-\infty}^{+\infty} (x'^{-}(t) + y'^{-}(t))dt + \sum a_i^{-} + \sum c_i^{-} \right] < 1 \tag{3.3}$$

where $x'^{+}(t)$, $y'^{+}(t)$, a_i^{+}, and c_i^{+} are the positive portions or values of the corresponding non-superscripted functions or numbers and $x'^{-}(t)$, $y'^{-}(t)$, a_i^{-}, and c_i^{-} are the negative portions or values of the corresponding non-superscripted functions or numbers. The magnitude of the piecewise continuous component of $x(t)$ is assumed to be less than $\ell \exp(ft)$ where ℓ and f are positive numbers.

Proof. The proof is identical with the proof of theorem 1.2 except for the handling of the $\int_0^{T_n} \dot{\sigma}(t)\, \phi(t)dt$ term. Because of condition a of the theorem, it is possible to express $\phi(\sigma,t)$ as

$$\phi(\sigma,t) = A\phi_m(\sigma) + \phi_2(\sigma,t) \tag{3.4}$$

where

$$|\phi_2(\sigma,t)| \leq (B-A)\ |\phi_m(\sigma)| \leq \frac{B-A}{A}\ |\phi(\sigma,t)| \tag{3.5}$$

Using this result, it is desired to show that

$$\int_0^{T_n} \dot{\sigma}(t)\phi_2(\sigma,t)dt + \frac{B-A}{2A} \int_0^{T_n} [\dot{\sigma}^2(t) + k^2\ \sigma^2(t)]dt \geq 0. \tag{3.6}$$

Since

$$|\dot{\sigma}(t)\ \phi_2(\sigma,t)| \leq \frac{B-A}{A} |\dot{\sigma}(t)\ k\ \sigma(t)|$$

$$\leq \frac{.5(B-A)}{A} [\dot{\sigma}^2(t) + k^2\ \sigma^2(t)] , \tag{3.7}$$

(3.6) holds. (1.43) in the proof of theorem 1.2 is replaced by

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt + \alpha A \int_0^{T_n} \dot{\sigma}^n(t)\ \phi_m{}^n(t)dt$$

$$+ \alpha \int_0^{T_n} \dot{\sigma}\ (t)\phi_2(\sigma,t)dt + .5\alpha \frac{(B-A)}{A} \int_0^{T_n} ([\dot{\sigma}^n(t)]^2 + [k^2\ \sigma^n(t)]^2)dt$$

$$+ E \int_0^{T_n} (\sigma^n(t) - \phi^n(t)/k)\ \phi^n(t)dt =$$

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma_\phi{}^n(t))\ \phi^n(t)dt + \alpha \int_0^{T_n} \dot{\sigma}_\phi{}^n(t)\ \phi^n(t)dt$$

$$+ .5\alpha \frac{(B-A)}{A} \int_0^{T_n} [\dot{\sigma}_\phi{}^n(t)]^2dt + .5\alpha\frac{(B-A)}{A} k^2 \int_0^{T_n} [\sigma_\phi{}^n(t)]^2dt$$

$$+ E \int_0^{T_n} (\sigma_\phi{}^n(t) - \phi^n(t)/k)\ \phi^n(t)dt +$$

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma_i{}^n(t))\ \phi^n(t)dt + \alpha \int_0^{T_n} \dot{\sigma}_i{}^n(t)\ \phi^n(t)dt$$

$$+ \frac{\alpha(B-A)}{A} \int_0^{T_n} \dot{\sigma}_i{}^n(t)\ \dot{\sigma}_\phi{}^n(t)dt + \frac{\alpha(B-A)}{A} k^2 \int_0^{T_n} \sigma_i{}^n(t)\sigma_\phi{}^n(t)dt + E \int_0^{T_n} \dot{\sigma}_i{}^n(t)\phi^n(t)dt$$

$$+ \frac{.5\alpha(B-A)}{A} \int_0^{T_n} [\dot{\sigma}_i{}^n(t)]^2dt + \frac{.5\alpha(B-A)}{A} k^2 \int_0^{T_n} [\sigma_i^n(t)]^2dt . \tag{3.8}$$

Before applying Parseval's theorem to the integrals on the right hand side of equation (3.8), the $\sigma_\phi{}^n(t)$ and $\dot{\sigma}_\phi{}^n(t)$ terms must be replaced by $\sigma_\phi{}^{n*}(t)$ and $\dot{\sigma}_\phi{}^{n*}(t)$, respectively and the upper limits on the integrals changed to ∞. The only new step required is by the $\int_o^{T_n} [\dot{\sigma}_\phi{}^n(t)]^2 dt$ and $\int_o^{T_n} [\sigma_\phi{}^n(t)]^2 dt$ terms.

Let $\dot{\sigma}_\phi{}^d(t) = \dot{\sigma}_\phi{}^{n*}(t) - \dot{\sigma}_\phi{}^n(t)$ and $\sigma_\phi{}^d = \sigma_\phi{}^{n*}(t) - \sigma_\phi{}^n(t)$ where $\dot{\sigma}_\phi{}^d(t)$ is that component of $\dot{\sigma}_\phi{}^{n*}(t)$ outside $(0,T_n)$ and $\sigma_\phi{}^d(t)$ is that component of $\sigma_\phi{}^{n*}(t)$ outside $(0,T_n)$. Then

$$\int_o^\infty [\dot{\sigma}_\phi{}^n(t)]^2\, dt = \int_o^\infty [\dot{\sigma}_\phi{}^{n*}(t)]^2 dt - \int_o^\infty [\dot{\sigma}_\phi{}^d(t)]^2 dt \tag{3.9}$$

and

$$\int_o^\infty [\sigma_\phi{}^n(t)]^2 dt = \int_o^\infty [\sigma_\phi{}^{n*}(t)]^2 dt - \int_o^\infty [\sigma_\phi{}^d(t)]^2 dt\ . \tag{3.10}$$

Using the convolution theorem together with straightforward bounding techniques gives

$$\int_o^\infty [\dot{\sigma}_\phi{}^d(t)]^2 dt \leq |\phi^n(t)|^2_{max} \int_{T_n}^\infty \left[\int_{t-T_n}^t |F^{-1}(j\omega G(j\omega))|\, d\lambda\right]^2 dt \tag{3.11}$$

and

$$\int_o^\infty [\sigma_\phi^d(t)]^2 dt \leq |\phi^n(t)|^2_{max} \int_{T_n}^\infty \left[\int_{t-T_n}^t |F^{-1}(G(j\omega))|\; d\lambda\right]^2 dt. \tag{3.12}$$

With (3.9) and (3.10) used on the modified right hand side of (3.8), the Aizerman and Gantmacher completing the square approach together with condition (3.1) gives a bound on all of the integrals on the modified right hand side except for

$$\int_0^\infty [x(t) * \sigma_\phi{}^d(t)]\phi^n(t)dt + \frac{.5\alpha(B-A)}{A} \int_0^\infty [\dot{\sigma}_\phi{}^d(t)]^2 dt +$$

$$\frac{.5\ (B-A)}{A} k^2 \int_0^\infty [\sigma_\phi{}^d(t)]^2 dt + \frac{.5\alpha(B-A)}{A} \int_0^{T_n} [\dot{\sigma}_i{}^n(t)]^2 dt +$$

$$\frac{.5\alpha(B-A)}{A} \int_0^{T_n} [\sigma_i{}^n(t)]^2 dt \ . \tag{3.13}$$

Using the result obtained for the first integral of (3.13) in Chapter I together with (3.11) and (3.12) gives that the left hand side of (3.8) is less than or equal to

$$M_1 + |\phi^n(t)|^2_{max} M_2 \tag{3.14}$$

where M_1 and M_2 are positive numbers independent of T_n. Using (3.6) and (3.7) gives

$$\int_0^{T_n} ((\delta(t) + x(t) + y(t)) * \sigma^n(t))\ \phi^n(t)dt + \alpha A\ \Phi_m(t)$$

$$\leq \alpha A\ \Phi_m(0) + M_1 + |\phi^n(t)|^2_{max} M_2 \tag{3.15}$$

where $\Phi_m(t) = \int_0^{\sigma(t)} \phi_m(\sigma)d\sigma$. The above inequality shows that $\sigma(t)$ is bounded and that $\int_0^{T_n} \sigma(t)\phi(t)dt$ is also bounded, thereby demonstrating asymptotic stability in the large. Q.E.D.

<u>Example 3.1</u>. Let $G(s) = \dfrac{(s+.0001)(s+.05)}{(s+.1)(s+1)^3}$. The problem is to find the characteristics of the time varying nonlinearity that is permitted if the system is to be asymptotically stable in the large. $G(j\omega)$ has a leading phase angle outside the $\pm 90^\circ$ band at low frequencies and a lagging angle outside this band at high frequencies. A convenient choice for $Z(s)$ is $(-s+.05)(s+1)/(-s+.1)$. $Z(s)\ G(s)$ is then $(-s^2 + .0025)(s + .0001)/(-s^2 + .01)(s + 1)^2$, the real part of which is non-negative for all ω. Also, since $Z(s) = s + 1.05 - .055/(-s + .1)$, both $x(t)$ and $y(t)$ are non-positive and $\int_{-\infty}^{+\infty} (|x(t)| + |y(t)|)dt = .524$. Therefore, from (3.3) it follows that $B/A < 1.91$. Next k is determined by working with (3.1) with $E = 0$. The largest allowed value is $k = 2.10$. Therefore, any continuous time varying nonlinearity with a monotone bounding function $\phi_m(\sigma)$ such that the B/A inequality is satisfied and having a linear bound with a slope less than 2.10 is permitted. An example of an allowed function is $\phi(\sigma,t) = p\sigma(1 + q\cos\omega_0 t)/(1 + |\sigma|)$, where $0 < p < 1.6$ and $0 < q < .312$. For this case $\phi_m(\sigma) = p\sigma/(1 + |\sigma|)$.

The next theorem gives a bound on the response for ϕ being a time varying nonlinearity.

Theorem 3.2. For the system of figure 1 excited by initial conditions let a and b of theorem 3.1 hold and let

c. $$\mathrm{Re}\, H(j\omega) = \mathrm{Re}[c(1 + dj\omega + X(j\omega) + Y(j\omega))\, G(j\omega - a) + E(G(j\omega - a) + 1/k) + da\, G(j\omega - a) + d\,\frac{(B-A)}{2A}\,(k^2 + a^2 + \omega^2)\,|G(j\omega - a)|^2] \geq \delta > 0 \qquad (3.16)$$

for all real ω where a is a positive number whose magnitude is less than the magnitude of the real part of the pole of G(s) closest to the $j\omega$ axis and c, d and E are positive numbers. x(t) and y(t) are composed of delayed impulses and a piecewise continuous function that satisfies $x(t)=0$ for $t > 0$, $y(t) = 0$ for $t < 0$ and $x(t) \leq 0$ for $t < 0$ and $y(t) \leq 0$ for $t > 0$. The magnitude of the piecewise continuous component of x(t) is assumed to be less than ℓ exp(ft) where ℓ and f are positive numbers and

$$\int_{-\infty}^{+\infty} e^{-a|t|}\,|x(t) + y(t)|\,dt < 1. \qquad (3.17)$$

Then, $$\Phi_m(T_n) \leq e^{-2aT_n}\left[\frac{\int_0^\infty m^2(t)\,dt}{4dA} + \Phi_m(0) + \frac{M(T_n)\,|\phi^n(t)|^2_{max}}{dA} + \frac{R(T_n)}{dA}\right] \qquad (3.18)$$

where $\Phi_m(T_n) = \int_0^{\sigma(T_n)} \phi_m(\sigma)\,d\sigma$ and $m(t) = F^{-1}[P(j\omega)\,Q(j\omega)]$ with

$$p(t) = e^{at}\,[(c + 2ad + E)\,\sigma_i^n(t) + d\,\dot{\sigma}_i^n(t)] + c\,[\sigma_i^n(t)e^{at} * (x(t) + y(t))]^n + \frac{d(B-A)}{A}\,e^{at}\,[\sigma_i^n(t) + k^2\,\sigma_i^n(t)].$$

$Q(j\omega)$ is defined by $1/\text{Re}\, H(j\omega) = Q(j\omega)\, Q(-j\omega)$,

$$M(T_n) = c \int_0^{T_n} e^{at} \int_{-\infty}^{t} e^{a(t-\lambda)} \int_{t-\lambda-T_n}^{t-\lambda} |g(\varepsilon)||x(\lambda)|\, d\varepsilon\, d\lambda\, dt$$

$$+ \frac{.5d(B-A)}{A} \int_{T_n}^{\infty} e^{2at} \left[\int_{t-T_n}^{t} |F^{-1}(j\omega\, G(j\omega))|\, d\lambda \right]^2 dt +$$

$$\frac{.5d(B-A)}{A} \int_{T_n}^{\infty} e^{2at} \left[\int_{t-T_n}^{t} |F^{-1}(G(j\omega)|\, d\lambda \right]^2 dt, \text{ and}$$

$$R(T_n) = \frac{.5d(B-A)}{A} \int_0^{T_n} e^{2at} \left([\sigma_1^{\,n}(t)]^2 + [\dot{\sigma}_1^{\,n}(t)]^2 \right) dt.$$

<u>Proof</u>. The proof of this theorem is similar to that of theorems 1,2, 2.4, and 3.1. A modification required for this case occurs for the $\int_0^{T_n} e^{2at}\, \dot{\sigma}(t)\, \phi(\sigma,t)dt$ term. It may be rewritten as

$$\int_0^{T_n} e^{2at}\, \dot{\sigma}(t)\, \phi(\sigma,t)dt = A \int_0^{T_n} e^{2at}\, \dot{\sigma}(t)\, \phi_m(\sigma(t))dt +$$

$$\int_0^{T_n} e^{2at}\, \dot{\sigma}(t)\, \phi_2(\sigma,t)dt. \tag{3.19}$$

Integration by parts gives for the first integral on the right hand side of (3.19)

$$Ae^{2aT_n}\, \Phi_m(\sigma) - A\, \Phi_m(\sigma(0)) - 2aA \int_0^{T_n} e^{2at}\, \Phi_m(t)dt. \tag{3.20}$$

The second integral on the right hand side of (3.19) is less than

$$\frac{.5(B-A)}{A} \int_{o}^{T_n} [\dot{\sigma}^2(t) + k^2 \sigma^2(t)]e^{2at}\, dt \tag{3.21}$$

Using these modifications together with the approaches already employed gives the proof of the theorem. Q.E.D.

The conditions of the theorems for the time varying case are a good deal more complicated than their time invariant counterparts; there appears to be no way of simplifying these results and still obtaining improved conditions for asymptotic stability.

B. Application to Sampled Data Systems

In this section the techniques of the foregoing work are used to derive an improved stability criterion for sampled data systems. To the authors' knowledge, the best results obtained thus far for the single nonlinearity system are due to Jury and Lee[20]. Their criterion includes that of Tsypkin[21] as a special case. For asymptotic stability in the large it is required that the following relationship be satisfied on the unit circle:

$$\mathrm{Re}\ G^*(z)\ [1 + q(z - 1)] + 1/K - K' \frac{|q|}{2}\ |(z - 1)G^*(z)|^2 \geq 0,$$

where $0 < \phi(\sigma)/\sigma < K$ and $|\frac{d\phi(\sigma)}{d\sigma}| < K'$. In the above inequality $(z - 1)$ is analogous to the $j\omega$ term in the Popov criterion. Theorem 3.3 given below permits an entire class of multipliers to be used.

a. A Theorem for Monotone Nonlinearities

Theorem 3.3. For the system shown in Figure 11 let the following hold:

a. $0 \leq d\phi(\sigma)/d\sigma \leq k_2$ where k_2 is a positive number, both $\phi(\sigma)$ and $\sigma - \phi(\sigma)/k_2 = 0$ only for $\sigma = \phi(\sigma) = 0$, and $d\phi(\sigma)/d\sigma$ is a continuous function of σ.

b. $G^*(z)$ is a rational function of z having all of its poles inside the unit circle and the corresponding time function $g(i)$ is zero for i negative. The numerator and the denominator of $G^*(z)$ are assumed to have no common factors outside or on the unit circle in the z plane.

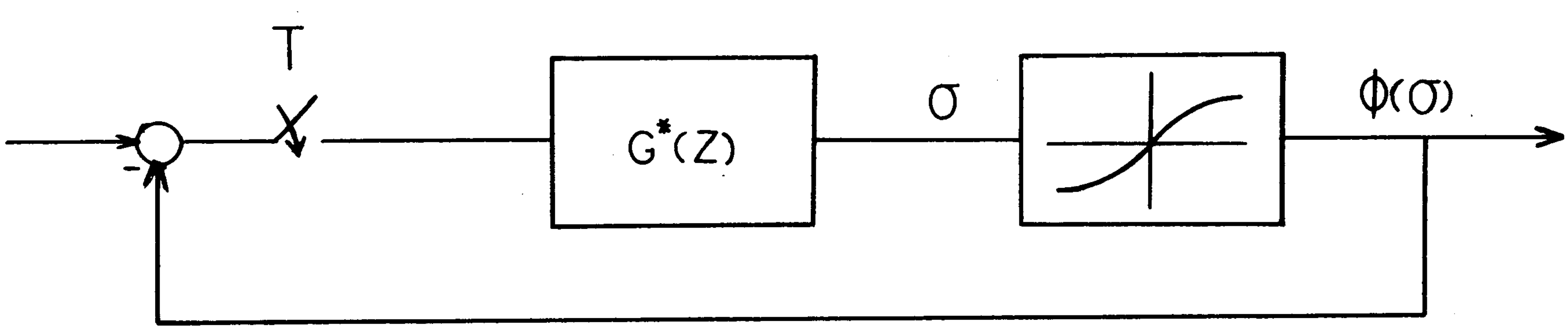

Figure 11. The Sampled Data System Under Consideration.

c. $\lim\limits_{|\sigma| \to \infty} (\sigma - \phi(\sigma)/k_2)\ \phi/|\phi(\sigma)|^2 = \infty.$

Then a sufficient condition for asymptotic stability in the large is that

$$\mathrm{Re}\ [R^*(z)\ (G^*(z) + 1/k_2)] \geq 0 \tag{3.22}$$

for $z = e^{j\omega T}$ for $0 \leq \omega \leq 2\pi$ where

$$R^*(z) = 1 + X^*(z) + Y^*(z)\ . \tag{3.23}$$

The time function $x(i) = 0$ for $i > 0$ and ≤ 0 for $i < 0$ while $y(i) = 0$ for $i < 0$ and ≤ 0 for $i > 0$. These functions must also satisfy

$$\sum_{i=-\infty}^{+\infty} (|x(i)| + |y(i)|) < 1. \tag{3.24}$$

The magnitude of $x(i)$ is less than ℓ exp (fi) where ℓ and f are positive numbers.

Corollary 1. In addition to the conditions of theorem 3.3, if $\phi(\sigma)$ is an odd monotone nonlinearity, that is, if $\phi(\sigma) = -\phi(-\sigma)$, the assertion of the theorem holds with $x(i)$ and $y(i)$ permitted to take on positive as well as negative values.

Corollary 2. If $G^*(z)$ has poles on the unit circle, $G^*(z)$ is required to be stable in the limit; that is for an arbitrarily small positive number ε, the roots of $1 + \varepsilon G^*(z)$ must all lie inside the unit circle. Also, the slope condition becomes $\geq \delta > 0$ where δ is an arbitrarily small positive number. The other conditions are unchanged except for (3.22) being $\geq \delta_1 > 0$.

Proof. First it will be shown that

$$\sum_{i=o}^{n} \phi^n(i)\ (\sigma^n(i) - \phi^n(i)/k_2) +$$

$$\sum_{i=o}^{n} \phi^n(i) \sum_{h=-\infty}^{i} [x(h) + y(h)][\sigma^n(i - h) - \phi^n(i - h)/k_2] =$$

$$c(n) \sum_{i=o}^{n} \phi^n(i)\ (\sigma^n(i) - \phi^n(i)/k_2) \tag{3.25}$$

where c(n) is a positive number. The second summation on the left hand side can be rewritten as

$$\sum_{i=o}^{n} \phi^n(i) \sum_{h=-\infty}^{o} x(h)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2] +$$

$$\sum_{i=o}^{n} \phi^n(i) \sum_{h=o}^{\infty} y(h)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2]. \tag{3.26}$$

Interchanging the order of summation gives

$$\sum_{h=-\infty}^{o} x(h) \sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2] +$$

$$\sum_{h=o}^{\infty} y(h) \sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2]\ . \tag{3.27}$$

Rewriting $\sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i - h) - \phi^n(i - h)/k_2]$ in terms of the positive and negative components of $\phi^n(i)$ and $\sigma^n(i - h) - \phi^n(i - h)/k_2$ and applying lemma 1 given at the end of Chapter 2 results in

$$\sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i-h) - \phi^n(i-h)/k_2] \leq$$

$$\sum_{i=o}^{n} \phi^n(i)\ [\sigma^n(i) - \phi^n(i)/k_2\ ] \tag{3.28}$$

Using (3.22) from the statement of the theorem together with (3.28) shows that (3.25) holds.

Letting $\sigma^n(i) = \sigma_\phi{}^n(i) + \sigma_i{}^n(i)$ in the left hand side of (3.25) gives for this side of the equation

$$\sum_{i=o}^{n} \phi^n(i)(\sigma_\phi{}^n(i) - \phi^n(i)/k_2) +$$

$$\sum_{i=o}^{n} \phi^n(i) \sum_{h=-\infty}^{i} [x(h) + y(h)][\sigma_\phi{}^n(i-h) - \phi^n(i-h)/k_2] +$$

$$\sum_{i=o}^{n} \phi^n(i)\ [\sigma_i{}^n(i) + \sum_{h=-\infty}^{i} [x(h) + y(h)]\ \sigma_i{}^n(i-h)] \tag{3.29}$$

Let $\sigma_\phi{}^n(i)$ be replaced by $\sigma_\phi{}^{n*}(i)$ where

$$\sigma_\phi{}^{n*}(i) = -Z^{-1}[G^*(z)\ Z[\phi^n(i)]].$$

This substitution can be made without changing the values of the summations in the first two summations of (3.29) except for the term involving x(i). Since x(i) is not zero for i < 0, the value of $\sigma_\phi{}^{n*}(i)$ for i > n will contribute to the result obtained by convolution. Therefore, the summation involving $x(i)$ is handled separately by making the substitution

$$\sigma_\phi{}^n(i) = \sigma_\phi{}^{n*}(i) - \sigma_\phi{}^d(i)$$

which gives for the total summation where the limits on i have been extended to $\pm\infty$,

$$\sum_{i=-\infty}^{+\infty} \phi^n(i)\ (\sigma_\phi{}^{n*}(i) - \phi^n(i)/k_2) +$$

$$\sum_{i=-\infty}^{+\infty} \phi^n(i) \sum_{h=-\infty}^{i} [x(h) + y(h)][\sigma_\phi{}^{n*}(i-h) - \phi^n(i-h)/k_2]$$

$$- \sum_{i=-\infty}^{+\infty} \phi^n(i) \sum_{h=-\infty}^{i} x(h)\ \sigma_\phi{}^d(i-h)\ . \tag{3.30}$$

With $\sigma_\phi{}^d(i) = \sum_{m=i-n}^{i} g(i-m)\ \phi^n(m)$, $i \geq n$, using the exponential character of $g(i)$ and $x(i)$ as in the proof of theorem 1.1 it can be shown

$$\left| \sum_{i=-\infty}^{+\infty} \phi^n(i) \sum_{h=-\infty}^{i} x(h)\ \sigma_\phi{}^d(i-h) \right| \leq M_1 |\phi^n(i)|^2_{max} \tag{3.31}$$

where M_1 is a positive number independent of n and $|\phi^n(i)|_{max}$ is the largest magnitude of $\phi^n(i)$ for $0 \leq i \leq n$. Applying the Liapunov-Parseval theorem to the first two summations of (3.30) gives

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} [1 + X^*(e^{j\omega T}) + Y^*(e^{j\omega T})][G^*(e^{j\omega T}) + 1/k_2]\ |Z[\phi^n(i)]|^2\ d\omega T \tag{3.32}$$

where T is the sampling period. Since the imaginary part of the integrand does not contribute to the final result, (3.32) may be rewritten as

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathrm{Re}[1 + X^*(e^{j\omega T}) + Y^*(e^{j\omega T})][G^*(e^{j\omega T}) + 1/k_2] \; |Z[\phi^n(i)]|^2 \, d\omega T \tag{3.33}$$

which is non-positive by (3.22). Combining (3.25) with (3.29), (3.31), and (3.33) gives

$$c(n) \sum_{i=o}^{n} \phi^n(i)(\sigma^n(i) - \phi^n(i)/k_2) \leq M_1|\phi^n(i)|^2_{max} +$$

$$\left| \sum_{i=o}^{n} \phi^n(i)[\sigma_i{}^n(i) + \sum_{h=-\infty}^{i} [x(h) + y(h)] \, \sigma_i{}^n(i - h)] \right|. \tag{3.34}$$

The second summation on the right hand side of (3.34) is less than or equal to

$$|\phi^n(i)|_{max} \sum_{i=o}^{\infty} \left|\sigma_i{}^n(i) + \sum_{h=-\infty}^{i} [x(h) + y(h)] \, \sigma_i{}^n(i - h)\right|$$

$$= |\phi^n(i)|_{max} \, M_2 \tag{3.35}$$

where M_2 is a positive number independent of n. Using (3.35) in (3.34) gives

$$c(n) \sum_{i=o}^{n} \phi^n(i)(\sigma^n(i) - \phi^n(i)/k_2) \leq M_1|\phi^n(i)|^2_{max} + M_2|\phi^n(i)|_{max}. \tag{3.36}$$

Let n be chosen such that $|\phi^n(i)|_{max}$ occurs at $i = n$. Using condition c of the statement of the theorem it follows that $\sigma^n(i)$ and $\phi^n(i)$ are bounded. Also, since the right hand side of (3.36) is independent of n, it follows that $\sigma^n(i)$ and $\phi^n(i)$ approach zero as i approaches infinity. Because of the assumptions on $G^*(z)$, it also follows that the other state variables of the system are also bounded and approach zero as $i \to \infty$. Therefore, the system is asymptotically stable in the large. Q.E.D.

The assertion of corollary 1 follows from the application of the lemma given at the end of chapter 2 to $|\sum_{i=o}^{n} \phi^n(i)(\sigma^n(i-h) - \phi^n(i-h)/k_2)|$ to get as a bound on this quantity $\sum_{i=o}^{n} \phi^n(i)(\sigma^n(i) - \phi^n(i)/k_2)$. The remainder of the proof is unchanged.

Corollary 2 follows from the transformation $\phi(\sigma) = \phi_1(\sigma) + \varepsilon\sigma$ and $G_1^*(z) = G^*(z)/1 + \varepsilon G^*(z)$ which results in a system that satisfies the conditions of the theorem.

An Allowed $R^*(z)$

Consider $\pi_i\left(\frac{z - b_i}{z - a_i}\right) \pi_h\left(\frac{z - c_h}{z - d_h}\right)$ with

$$0 < a_1 < b_1 < a_2 < b_2 \quad \ldots < a_n < b_n < 1$$

$$1 < c_1 < d_1 < c_2 < d_2 \quad \ldots < c_m < d_m.$$

Expansion of this function in a partial fraction expansion gives where A_i and B_h are positive numbers

$$1 - \sum_i \frac{A_i}{z-a_i} + \sum_h \frac{B_h}{z-d_h}$$

$$= 1 - \sum_i \frac{A_i z^{-1}}{1 - a_i z^{-1}} - \sum_h \frac{B_h/d_h}{1 - z/d_h}$$

from which it is seen that both x(i) and y(i) are non-positive. The total area

$$\sum_{i=-\infty}^{\infty} |x(i)| + |y(i)| = 1 - \pi_i\left(\frac{1 - b_i}{1 - a_i}\right) \pi_h \left(\frac{1 - c_h}{1 - d_h}\right) < 1.$$

Therefore, this function is an allowed one for the general monotone nonlinearity.

Example 3.2. Let $G^*(z) = \frac{3.6}{z-.9} - \frac{1.2}{z - .3}$ and $0 < k_2 < 1$.

$$G^*(z) + 1/k_2 = \left(\frac{z + .3}{z - .3}\right)\left(\frac{z + .9}{z - .9}\right). \quad \text{Let } R^*(z) = \frac{z - .3}{z + .3}.$$

Expressing this function in the time domain gives

$$R^*(z) = 1 - .6z^{-1} + 2(.3)^2 z^{-2} - 2(.3)^3 z^{-3} + \ldots$$

from which it is seen that y(t) takes on both positive and negative values and that the summation of the magnitude is 6/7. Therefore, this $R^*(z)$ may be used with symmetrical monotone nonlinearities.

$R^*(z)\ (G^*(z) + 1) = (z + .9)/(z - .9)$. The angle of this product on the unit circle is $- \tan^{-1}(9.48 \sin \omega T)$. Therefore, the criterion is satisfied and the system is asymptotically stable in the large for the given range of k_2.

C. The Multiple Nonlinearity Problem

Application of the by now standard approach gives the following theorem for a system having a number of nonlinearities.

Theorem 3.4.

For a continuous system with i nonlinearities let the following conditions hold:

a. $0 \leq d\,\phi_i(\sigma_i)/d\sigma_i \leq k_{2i}$ where k_{2i} is a positive number, both $\phi_i(\sigma_i)$ and $\sigma_i - \phi_i(\sigma_i)/k_{2i} = 0$ only for $\sigma_i = \phi_i(\sigma_i) = 0$, and $d\phi_i(\sigma_i)/d\sigma_i$ is a continuous function of σ_i.

b. The transfer function $-\,G_{ij}(s)$ relating $F(\sigma_i(t))$ to $F(\phi_j(t))$ is a rational function of s with the number of zeros at least one less than the number of poles and with all of the poles in the left half s plane.

c. $\lim\limits_{|\sigma_i| \to \infty} \int_o^{\sigma_i} \phi_i(\sigma_i)d\sigma_i/|\phi_i(\sigma_i)|^2 = \infty.$

Then a sufficient condition for asymptotic stability in the large is that the Hermitian matrix $H(j\omega)$ be positive semi-definite where

$$H(j\omega) = \begin{bmatrix} h_{11}(j\omega) & h_{12}(j\omega) \;\cdots\cdots & \\ h_{21}(j\omega) & h_{22}(j\omega) \;\cdots\cdots & \\ & & \\ & & h_{nn}(j\omega) \end{bmatrix}$$

where $h_{ii}(j\omega) = \text{Re}\, Z_i(j\omega)\, [(G_{ii}(j\omega) + 1/k_{2i})]$ and

$$h_{ij}(j\omega) = \frac{1}{2}\, [Z_i(j\omega)\, G_{ij}(j\omega) + \overline{Z_j(j\omega)\, G_{ji}(j\omega)}] \text{ for } i < j$$

and $\quad h_{ij}(j\omega) = \overline{h_{ji}(j\omega)}$ for $i > j$.

$$Z_i(j\omega) = 1 + \alpha_i j\omega + X_i(j\omega) + Y_i(j\omega)$$

where α_i is a positive number, $x_i(t) = 0$ for $t > 0$ and $y_i(t) = 0$ for $t < 0$ with both of these functions being non-positive and consisting of the sum of a piecewise continuous function which is Fourier transformable and shifted impulse functions that satisfy

$$\int_{-\infty}^{+\infty} (|x_i(t)| + |y_i(t)|)dt < 1.$$

Corollary 1. In addition to the conditions of theorem 1, if $\phi_i(\sigma_i)$ is an odd monotone nonlinearity, the assertion of the theorem holds with $x_i(t)$ and $y_i(t)$ being permitted to take on positive as well as negative values.

Proof. The proof of this theorem parallels that of theorem 1.1 but instead of working with one function there are n functions. The only variation occurs after applying Parseval's theorem. The quadratic form that is obtained is associated with a Hermitian matrix which is required to be positive definite. After applying this condition, the inequality given below is obtained.

$$\sum_{i=1}^{n} c_i (T_n) \int_o^{T_n} [\sigma_i^n(t) - \phi_i^n(t)/k_2]\, \phi_i^n(t)dt +$$

$$\sum_{i=1}^{n} \alpha_i \quad \Phi_i(T_n) \leq \sum_{i=1}^{n} M_{1i} \, |\phi_i{}^n(t)|^2_{max} + \sum_{i=1}^{n} M_{2i} \, |\phi_i{}^n(t)|_{max}$$

$$+ \sum_{i=1}^{n} \alpha_i \quad \Phi_i(0).$$

The reasoning of theorem 1 leads to the conclusion that all of these variables are bounded and approach zero as $t \to \infty$.

Example 3.3. This example was considered by Ibrahim and Rekasius[22]. The system consists of two nonlinearities connected in a single loop with linear elements in between. $G_1(s) = 1/(s+5)$ and $G_2(s) = (s+1)/(s+2)(s+3)$. For this case, $G_{11}(s) = G_{22}(s) = 0$, $G_{12}(s) = -1/(s+5)$ and $G_{21}(s) = (s+1)/(s+2)(s+3)$. The + sign for $G_{21}(s)$ is due to the feedback being negative. It is assumed that both nonlinearities are continuous monotone functions.

$$|H(j\omega)| = \frac{ReZ_1}{k_{21}}\frac{ReZ_2}{k_{22}} - \frac{1}{4}\left|\frac{Z_1(j\omega)}{(j\omega + 5)} - \frac{Z_2(-j\omega) \quad (-j\omega + 1)}{(-j\omega + 2)(-j\omega + 3)}\right|^2 \geq 0.$$

If asymptotic stability in the large is to be shown for $0 < k_{21} < \infty$ and $0 < k_{22} < \infty$, two functions $Z_1(j\omega)$ and $Z_2(j\omega)$ must be found such that the quantity inside the magnitude squared brackets is zero. This requires that

$$\frac{Z_1(j\omega)}{(j\omega+5)} = \frac{Z_2(-j\omega)(-j\omega + 1)}{(-j\omega+ 2)(-j\omega + 3)} .$$

Let $Z_2(j\omega) = \dfrac{(-j\omega + 1)(j\omega + 3)}{(-j\omega + 2)} = j\omega + 4 - \dfrac{5}{-j\omega + 2}$ and

$$Z_1(j\omega) = \frac{(j\omega + 5)(-j\omega + 1)(j\omega + 1)}{(-j\omega + 2)(j\omega + 2)} = j\omega + 5 - \frac{\frac{9}{4}}{j\omega + 2} - \frac{\frac{21}{4}}{-j\omega + 2}.$$

A check of the integral magnitude condition for these two functions reveals that they are allowed functions for general monotone nonlinearities. Substitution of these expressions gives $(\omega^2 + 1)/(\omega^2 + 4)$ on both sides of the equation. Therefore, it has been shown that the given system is asymptotically stable in the large for monotone nonlinearities having arbitrarily large slopes. In [22], asymptotic stability was shown for $k_{21} = k_{22} = 6$.

Conclusion

This chapter has applied the method of chapters 1 and 2 to get improved theorems for a time varying nonlinearity, for a sampled data system, and for a system with a number of nonlinearities. In order to show how useful these theorems are, it will be necessary to consider a number of different examples for each case.

VI. CONCLUSION

From the conclusions given at the end of each chapter it is apparent that additional research in the area of time-frequency domain stability criteria should be worth-while. In particular, the problem of the closeness of the stability results to the actual absolute stability boundary is an important one for future study.

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